The Value of Monitoring in Dynamic Screening

Dhruva Bhaskar*

NYU

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Abstract

This paper studies the use of endogenous monitoring as a screening device in long term relationships with severe adverse selection and limited commitment. A principal can hire an agent with hidden ability but strictly prefers his outside option to employing an unskilled agent. As unskilled agents find working more costly, the optimal monitoring policy screens agents by providing unskilled agents with incentives to shirk while skilled agents exert effort. Once the agent is monitored and their ability identified, the principal fires the agent if unskilled, and continues the relationship otherwise. The optimal contract involves inefficiently high monitoring by the principal after he learns that the agent is skilled. Excessive monitoring of high ability workers makes pretending to be skilled less attractive to the unskilled and allows faster screening. This highlights the key trade-off faced by the principal: providing cost-efficient incentives for skilled workers versus dismissing unskilled workers sooner. Monitoring stochastically declines over the course of the relationship.

*Email: dhruva.bhaskar@nyu.edu

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1 Introduction

This paper studies the use of endogenous monitoring as a screening device in long-term relationships with adverse selection. In many contracting relationships, adverse selection is a severe problem and there are certain kinds of people that are simply not profitable to employ. Consider a principal who must decide whether or not to continue a relationship with an agent who has private information regarding their ability (type). Only high types are worth employing, while both high and low types find employment desirable. Furthermore, the principal lacks performance measures so cannot condition his decision on what the agent produces.

The principal has limited commitment in that he cannot commit whether or not to fire the agent. Therefore he cannot screen by offering separate contracts. He can, however, monitor the agent's actions on the job and commit to the frequency with which he does this. Low types have a higher cost of exerting effort than high types. Monitoring is used to screen the low type initially. The principal designs the optimal contract to leave low types with incentives to take bad actions, despite the chance of being caught and fired. After screening, monitoring is used to incentivise effort from high types, as well as to make the latter stage of the contract less attractive to low types. The interaction of these different roles leads to an unusual trade-off for the principal. When moral hazard is the primary concern, the principal's priority is to monitor as cost-effectively as possible while providing sufficient incentives for effort. With the introduction of adverse selection, the principal actually distorts the efficiency of incentives provided to high types in order to be able to screen the low types faster. The optimal contract has inefficiently high levels of monitoring after screening. The frequency of monitoring monotonically decreases over the duration of the relationship.

As a motivating example, think of an investor who funds a startup that owns a new technology and has private information about its viability. The investment must be made early in order to reap any potential benefit, while the true quality of the technology may not be observed by an investor a long time - it can take years for a technology to be developed and reach profitability. This creates incentives for potential candidates for funds to misrepresent themselves. There have been numerous cases of startups with suspected fraudulent technology that raised millions of dollars of early investment. By the time Theranos - a company claiming to own a new technology for quick and cheap blood tests - was accused of fraud in 2015, they had already raised hundreds of millions of dollars in funds from investors. The doubts raised about the viability of the company's technology saw the value of the company plummet from nine billion dollars to 800 million dollars, and investors seeking lawsuits.[1] Investors monitor companies by periodically evaluating their activities in order to learn their quality at an earlier stage. [2] High quality startups find it in their interest to invest funds into the company as this benefits them directly. Fraudulent startups are more inclined to divert funds as they know that the company will not be profitable. However, they may find it worthwhile to invest spend money on the company in an attempt to appear credible and receive further funding.[3]

Consider also the monitoring of employees by an organisation that requires high-skilled labour, or the requirements for scientists to submit grant proposals when applying for funding from the NSF. In the former, it may be impossible to measure one individual's performance accurately when output is the aggregate of many agents' efforts. In the latter, scientific breakthroughs occur infrequently and over many years. Why do we observe researchers with excellent track records having to reapply for funding on a regular basis? The results suggest a possible explanation that asking established people to jump through hoops helps to deter fraud by unskilled scientists.[4]

The preference alignment between high types and the principal will depend on the application in question. Owners of a viable technology have a share in future profits so there may be no moral hazard with regard to the actions of the high type. In other situations it can be that though high types find effort less costly, they would rather shirk if the opportunity arises: employees in an organisation that are good at their job may still shirk responsibilities if incentives are not provided. I analyse both these situations: the first corresponds to one of pure adverse selection, the second to one of adverse selection and moral hazard. While

[1]For a summary of the events surrounding Theranos, see Popken (2016) and Weaver (2015, 2016).

[2]This is a well-documented aspect of the relationship between venture capitalists and their investments. See Gompers and Lerner (2004) for a detailed summary. Gompers (1995) finds that "the evidence suggests that venture capitalists are concerned about the lack of entrepreneurial incentive to terminate projects when it becomes clear that projects will fail...By gathering information, venture capitalists determine whether projects are likely to succeed and continue funding only those that have high potential".

[3]Indeed, such deception has been documented. For instance, the *Wall Street Journal* (Weaver, Carreyrou and Siconolfi (2016)) states that "major investors... toured the Theranos's facilities and were shown versions of the company's proprietary devices [prior to investment]". Investments were made following such monitoring activities, but now it appears that investors are likely to lose nearly all of their money due to the failure of the company's technology.

[4]Misconduct in scientific research is a common problem and there have been many instances of researchers submitting false reports and proposals. For example, in 2016, a medical researcher from the University of Michigan was found by the Office of Research Integrity (ORI) to have made up more than 70 experiments on heart cells. These were included in three grant applications to the National Institute of Health (NIH) over six years.

differences arise in the details of the optimal contract, the central idea - the manner in which monitoring is used for screening, and the trade-off the principal faces - is the same.

The optimal contract consists of two phases. We can think of Phase 1 as the screening phase, in which the principal sets a trap for the low type. Phase 1 lasts as long as the principal has not monitored in the past. The principal monitors with a constant probability every period, while the low type shirks, and the high type exerts effort. The first time that the principal actually monitors, he learns the agent's type, fires the agent if he was shirking, and begins Phase 2 of the contract if effort is observed. To make shirking in Phase 1 incentive compatible for the low type, the principal must monitor with a sufficiently low probability. In particular it needs to be low relative to the value that the low type could get from deviating and entering Phase 2. For instance, in the case of pure adverse selection, the efficient thing to do in Phase 2 is to stop monitoring as the high type requires no incentives. If the principal does this, the low type has a strong incentive to pretend to be the high type unless monitoring in Phase 1 is very low. The principal would like to identify the low type as quickly as possible in Phase 1 and minimise costs of monitoring in Phase 2. There is some tension between these two objectives. The optimal contract balances this trade-off, and involves distortion to Phase 2 of the contract to screen faster in Phase 1.

The distortion involves inefficiently high monitoring at the beginning of Phase 2. Wasteful monitoring occurs early on, and the relationship converges to the Pareto frontier. In the case of pure adverse selection, the distortions involve monitoring even after the principal knows that the agent is the high type. The frequency of monitoring declines over time and monitoring eventually ends forever. When there is also moral hazard, the high type's incentive to exert effort is slack at the beginning of Phase 2. This is inefficient as there is more monitoring than needed to provide incentives. The frequency of monitoring declines each time the principal monitors. The relationship eventually reaches the Pareto frontier, starting at the principal-optimal point, and moving along the Pareto frontier over time until it arrives at the agent-best point. After this the high type permanently enters "semi-retirement" where he works some of the time and is allowed to shirk for blocks of time (during which the principal does not monitor). This backloading of agent rewards is a robust phenomenon which appears in many contracting environments. Pushing agent payoffs into the future allows the principal to use lower levels of monitoring early on in the relationship and thus economise on monitoring costs.

The optimal contract exhibits decreasing monitoring frequency over time, until monitoring either eventually ends, or becomes relatively infrequent. These dynamics are supported by empirical evidence from Sapienza (1996). Companies that have spent fewer years in a venture capitalist's portfolio face significantly greater monitoring than older investments. Higher uncertainty about the success of the venture results in more monitoring, with early stage ventures receiving more involvement from investors.[5] Duffner et. al (2009) identify the level of monitoring of firms as a substitute for trust in the relationship, suggesting that monitoring declines as trust grows. On the equilibrium path, I find that monitoring, learning and firing occurs. This is consistent with empirical evidence of monitoring by venture capitalists and termination of projects which are unlikely to succeed.[6]

The use of endogenous monitoring has been well studied in the context of moral hazard. Monitoring of actions is used to incentivise agents to exert costly effort, and it must be that the level of monitoring and punishment for shirking is sufficiently high. In the context of adverse selection, however, monitoring is also used as a learning tool. Monitoring, or sufficient lack of it, is used to set a trap to entice undesirable agents into shirking. In particular, the principal *wants* to monitor and see the agent shirking, while in the context of moral hazard, the principal would rather not monitor conditional on the agent taking the desired action. This means that the principal's commitment power is binding in the opposite direction when there is adverse selection as compared to moral hazard. In both cases, commitment makes it feasible to monitor randomly when the agent is taking a pure action. With adverse selection it allows the principal to *not monitor* with probability one when the agent is shirking and thus set an effective trap. With moral hazard it allows the principal to monitor with positive probability when the agent is working. The contribution of this paper is to study the use of monitoring as a learning tool and the interaction of this with the well-understood use in the context of moral hazard.

The principal's ability to commit to monitoring is important in the optimal contract. A natural question that arises is how effective is monitoring in this environment when the principal cannot commit to the monitoring policy? I focus on principal-best equilibria of the relationship. In general, without commitment, endogenous monitoring has limited value for the principal. In the case of pure adverse selection, there is essentially a unique equilibrium which consists of a war of attrition between the principal and the low type. Since monitoring is costly, if the principal becomes sufficiently convinced that the agent is the high type, he will stop monitoring forever. Therefore the low type wants to build a reputation for being the high type. The equilibrium involves mixing by both players until either the agent is fired

[5] The results are from a survey of venture capitalists from the USA and Europe.

[6] See Gompers (1995).

or the principal's belief crosses a threshold. Both players strictly mix except possibly at time zero, if the principal is initially very pessimistic about the agent. In that case, at time zero, monitoring is a strict best response while the agent shirks with positive probability.[7] Other than that, not monitoring is always a best response for the principal, hence monitoring is of value only if the principal is sufficiently pessimistic about the agent initially. The principal's monitoring probability is decreasing in the agent's reputation. When there is also moral hazard, the best equilibrium for the principal involves monitoring with probability one, while the high type exerting effort in every period, supported by a grim trigger punishment (in which monitoring stops and the high type begins shirking forever, so the principal fires immediately) between the principal and the high type if either deviates. The low type does not find it worthwhile to exert effort and is screened immediately. However, this is extremely costly for the principal since he has to monitor all the time. As monitoring has to be a best response, the principal cannot benefit from stochastic monitoring.

The formal model in the paper does not allow for monetary transfers between the principal and the agent. For the purposes of screening, allowing for self-enforcing transfers with limited liability (from the principal to the agent) does not change anything as the principal cannot credibly pay the low type to reveal himself. In the pure adverse selection model, self-enforcing transfers will never be used in the optimal contract. Furthermore, depending on the setting, such payments may be more or less plausible. In section 6, I discuss what role transfers can play and in which applications they are more likely to be available. The rest of the paper is organised as follows. Section 3 presents the model and a discussion of the limited commitment assumption. Section 4 analyses the pure adverse selection case, and section 5 the case of moral hazard and adverse selection.

2 Related Literature

This paper uses elements from a number of different strands of the literature on contracting. The central feature of a principal attempting to learn about an agent's unknown ability is in the spirit of Holmström (1999), and the low type of agent's desire to convince the principal that he is the high type is reminiscent of agent signal-jamming when faced with career concerns. Incentivising agents to take separate actions in order to screen is in the vein

[7]This is a common feature of a war of attrition. The principal being pessimistic is akin to the low type being weak in a standard war of attrition. Therefore the low type needs to "concede" with some probability at time zero, after which the equilibrium moves to the balanced mixing path.

of the literature on signalling (e.g. Spence (1973)). Endogenous (and costly) monitoring is a feature that appears in the literature on costly state verification and in the literature on repeated games with costly monitoring (Ben-Porath and Kahneman (2003)).

There is a close connection to the literature on costly state verification (Townsend (1979) and audit models (e.g. Reinganum and Wilde (1981)) in which the principal can monitor agent compliance at some cost. The focus in most of this literature, is on the use of monitoring to enforce high effort by agents.[8] This is the more traditional and well-understood use of monitoring. Ichino and Muehlheusser (2008) and Sami (2009) are, to the best of my knowledge, the only other papers to identify the role that monitoring can play for screening in the presence of adverse selection. These papers considers a model with one period of monitoring in which monitoring is used to screen the agent's type. After the first period, the principal chooses to fire or hire the agent permanently. In this paper, monitoring is used to incentivise shirking from low types initially and make the latter phase of the contract less attractive for them, as well as to incentivise high types to work. In the separate roles, monitoring works quite differently since sufficiently high monitoring is required to enforce good behaviour or prevent successful shirking, but sufficiently low monitoring is needed to incentivise bad behaviour. The interaction of these different roles leads to an unusual trade-off for the principal. When moral hazard is the primary concern, the principal's priority is to monitor as cost-effectively as possible while providing sufficient incentives for effort. With the introduction of adverse selection, the principal actually distorts the efficiency of incentives provided for effort in order to be able to screen the agent faster.

There is a large literature on dynamic contracts with adverse selection and moral hazard, especially in the finance literature. Most of the papers in this literature take the monitoring structure as given but allow for a signal of the output that is generated by the agent's actions (See, for instance, Gershkov and Perry (2012), DeMarzo and Sannikov (2007 and 2016)). In these models, the principal makes inferences about the agent's type by observing the quality of output that is generated. The agent is able to bias the principal's beliefs by deviating from what he is supposed to do. Furthermore, typically in these models, having the exert higher levels of effort is more informative about the agent's quality so the principal wants to incentivise effort on the agent's part to learn. In this paper, there is no signal of output and the principal's information is endogenously generated by the choice of monitoring. Furthermore, the principal wants to separate the types by incentivising them to take different actions, as opposed to take similar but costly actions. Therefore the kind of incentives that

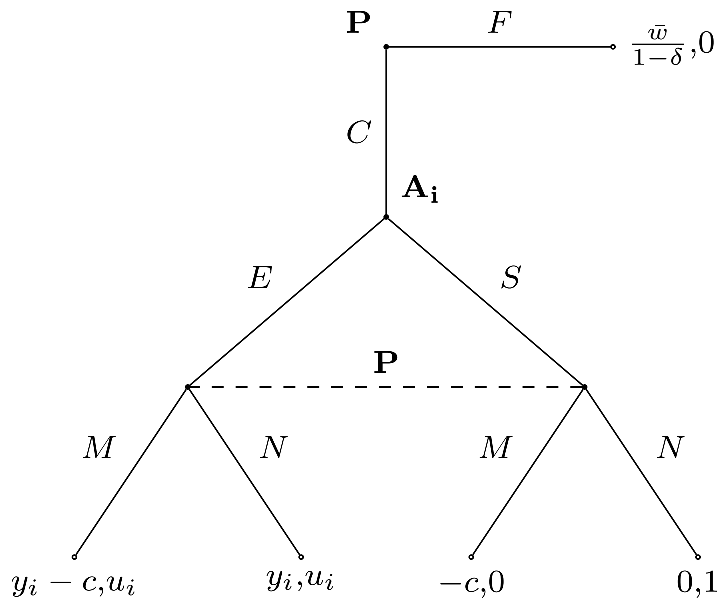
[8] For instance, Piskorski and Westerfield (2016) considers dynamic moral hazard with costly monitoring.

need to be provided to each type are starkly different.

The results when the principal has no commitment to monitoring in the case of pure adverse selection are closely related to Baliga and Ely (2016) in which a principal can use costly torture on an agent who may be informed or uninformed about a terrorist attack in the future. The paper is in continuous time, with the principal able to commit to fixed period lengths of torture. Their result that the value of torture goes to zero as the period length shrinks reminiscent of the result here that monitoring can have no value in equilibrium, but the reasoning behind the two results is quite different. In this model, this comes from the necessity of the principal's indifference when monitoring, so the principal weakly prefers to never monitor. In their model, there is positive value from a period of torture, but since there can only be finitely many periods of torture before the principal stops, the total value of torture goes to zero in the limit. My results connect strongly to the literature on reputation in repeated games (see Kreps and Wilson (1982), Milgrom and Roberts (1982), Fudenberg and Levine (1992)) with two long-lived players. Much of the of the reputation literature is concerned with the long-run reputation of players and bounds on the payoffs of the informed player. Several features of the model here mean that many of the typical reputation results do not apply - there is costly perfect monitoring, and the uninformed player only observes signals in periods that he monitors. This naturally limits the amount of information revealed in equilibrium, since the principal stops monitoring once the belief crosses a threshold. The mixing structure of the equilibrium is similar to that of a war of attrition such as arises in timing games or concession games. Hendricks, Weiss and Wilson (1988), Ordover and Rubinstein (1986) and Abreu and Gul (2000) exhibit the properties that appear in the unique equilibrium here. The gains from monitoring which occur at time zero correspond to a concession by a weak player at time zero in a standard war of attrition.

3 Model

There are two infinitely lived players, a principal (P) and an agent (A) who is employed by the principal. The agent has a privately known type $i \in \{H, L\}$, high or low. Time is discrete, starting in period 0. The stage game shown in Figure 1 is played in every period. In the stage game, the principal first decides to fire the agent (F) or continue (C) the relationship. Firing ends the game. If the principal continues, then both players moves simultaneously. The agent can either choose to exert effort (E) or shirk (S). The principal can either monitor



(M) or not monitor (N) the agent's actions. If he monitors, he perfectly observes what action the agent took. The principal's monitoring decision becomes common knowledge afterwards.

If the principal fires the agent, both players receive their outside option forever. The principal has an outside option of $\bar{w} \in (0, 1)$ and both types of agent have an outside option of 0. Both types of agent get a payoff of 1 if they shirk and are not monitored, and 0 if they shirk and are monitored. The high type of agent receives a payoff of u_H from effort, and the low type a payoff of u_L, with $u_H > 0 > u_L$.[9] The low type finds effort more costly than the high type. If $u_H > 1$ then the high type and the principal have perfectly aligned preferences. If $u_H < 1$ then there is a preference misalignment and the high type needs to be incentivised to exert effort. The principal receives an *unobservable* payoff from the agent's actions. For example, the profitability of a startup will not be observed for many years; output in an organisation is the aggregate of many agents' work and it is not possible to disentangle one individual's contribution to this. [10] When the agent exerts effort, this payoff is $y_H = 1$ if the agent is the high type, and $y_L = 0$ if he is the low type. It is 0 if the agent shirks.[11] Monitoring costs

[9]It is important that $u_H > 0$ so that it is feasible for the principal to have a profitable relationship with the high type. For many of the results, all that is needed qualitatively is that $u_L < u_H$. I assume $u_L < 0$ for simplicity.

[10]The unobservability of payoffs is a standard assumption in dynamic games of incomplete information. I believe that most of the qualitative features of the results do not depend on complete unobservability of payoffs but only require that signals of output are sufficiently noisy.

[11]I assume that $y_L = 0$ and that shirking gives the principal a payoff of 0 regardless of whether the agent

the principal $c > 0$.

The principal prefers effort from the high type to his outside option, but employing the low type is strictly worse than the outside option regardless of what action the low type takes.

Both players discount the future by $\delta \in (0,1)$ and payoffs are normalised by $(1-\delta)$. The principal's initial belief that the agent is the high type is $p_0 \in (0,1)$. I make the following assumptions:
(A1) $1 - c > \bar{w}$.
(A2) $(1-\delta)u_L + \delta \max\{u_H, 1\} > 0$.
A1 says that the cost of monitoring is not very high: always monitoring as long as the high exerts effort is better for the principal than his outside option. A2 makes the game non-trivial by ensuring that effort is not a strictly dominated action for the low type: he is willing to exert effort today if the future payoff is sufficiently high.

A (public) history of the game, h^t, at the start of period $t \geq 1$ is a t long sequence of the public events that can occur in stage game. These are that the principal did not monitor (N), the principal monitored and observed the agent exerting effort (E), or the principal monitored and observed the agent shirking (S). Histories can be coded in this way since the game ends if the principal fires the agent, and if the principal does not monitor, he observes nothing. Formally, $h^t \in \{N, E, S\}^t$. I define the initial history h^0 to be the null set. Let $\mathcal{H}$ be the set of all histories, and h^tN, h^tE and h^tS denote the concatenation of history h^t with the respective public events.

A strategy for the principal is a pair of maps, $d : \mathcal{H} \to [0,1]$ and $m : \mathcal{H} \to [0,1]$, where $d(h^t)$ is the probability of continuing the relationship at h^t, and $m(h^t)$ is the monitoring probability at h^t. A public strategy for agent type i is a map $s_i : \mathcal{H} \to [0,1]$ where $s_i(h^t)$ is the probability of shirking at h^t. I focus on agent strategies that are measurable with respect to the public history.[12]

Let $\sigma := \{d, m, s_H, s_L\}$ be a strategy profile. Given a profile σ, define the principal's belief that the agent is the high type at any history as the map $p : \mathcal{H} \to [0,1]$, with $p(\emptyset) := p_0$ and $p(h^t)$ defined according to Bayes' rule wherever possible. Any profile σ induces a probability

is caught for simplicity of exposition. The results of the paper will still hold if we allow $y_L < \bar{w}$, so the low type is not worth employing, and if the principal receives a payoff $w_S < \bar{w}$ when he catches the agent shirking.

[12]Note that the agent also has private histories of his own actions. The focus on public strategies is without loss of generality as far as equilibrium payoffs are concerned, since the principal has no private history and therefore can only play a public strategy. By standard results, for any strategy of the principal, the agent has a public strategy as a best reply.

distribution over histories, and let this be $\mathbb{P}^\sigma(h^t)$.

Define the principal's expected discounted sum of payoffs from a profile σ as

$$W(\sigma) :=$$
$$(1-\delta)\sum_{t=0}^{\infty}\sum_{h^t\in\mathcal{H}}\delta^t\mathbb{P}^\sigma(h^t)\left[\left(1-d\left(h^t\right)\right)\frac{\bar{w}}{1-\delta}+d\left(h^t\right)\left(p(h^t)\left(1-s_H(h^t)\right).1-m\left(h^t\right)c\right)\right]$$

Define agent type i's expected discounted sum of payoffs from a profile σ as

$$V_i(\sigma) := (1-\delta)\sum_{t=0}^{\infty}\sum_{h^t\in\mathcal{H}}\delta^t\mathbb{P}^\sigma(h^t)d(h^t)\left(s_i(h^t)(1-m(h^t)).1+\left(1-s_i(h^t)\right)u_i\right)$$

$W(\sigma)|_h$ and $V_i(\sigma)|_h$ denotes the expected payoffs of players from history h onwards.

3.1 Contracts with Limited Commitment

I assume that the principal has limited commitment to policies. In particular, he cannot commit to the firing decision d, but he can commit to the monitoring policy m.

Definition 1. *A contract is a profile $\sigma = \{d, m, s_H, s_L\}$ such that*
1) s_H and s_L are incentive compatible for each type of agent at every history:

$$V_i(\sigma)_h \geq V_i(s_i', \sigma_{-i})|_h$$

for all s_i', for all h, $i \in \{H, L\}$.
2) The principal's firing decision is optimal at every history:

$$W(\sigma)|_h \geq W(d', m, s_H, s_L)|_h$$

for all d', for all h.

Any firing decision must be a best-response for the principal. Importantly, if the principal learns that the agent is the low type, he will always fire the agent because the value of any relationship with the low type is inferior to the principal's outside option. A consequence of the limited commitment to firing means that whatever other commitment power the principal has, he cannot credibly offer separate contracts to screen the types of agent, since there is no contract that the principal can credibly offer the low type which doesn't involve immediate firing. This means that the principal must monitor and observe the actions taken by the agent to learn about the agent's type. The principal has full commitment to the monitoring policy as long as the relationship lasts. This means that at time zero, the principal specifies the monitoring probability at every history.

To understand the asymmetry in the principal's commitment power, it is important to think about the problem in the wider context of the settings I have described. While the model focuses on the principal's interaction with a single agent, in reality there will be many agents that a principal employs - a portfolio of companies, or a workforce of employees. Thinking of this, we can interpret the asymmetry in a number of ways.

First, a common explanation given for commitment to a particular form of action is that the principal finds it worthwhile to build a reputation for being able to commit to certain actions: the costs and benefits are such that it is worth building a reputation for commitment to monitoring, but not to the firing decision. This can occur, say if the principal does not employ very large numbers, and the amounts of money involved for in employment are high relative to the costs of monitoring.

Second, a key difference between monitoring and the firing decision is the kinds of justification one might need to provide for each one. Monitoring may be a routine procedure which the principal can choose as he sees fit, without justification. On the other hand, to fire an agent, just cause for dismissal needs to be provided: for instance if it is evident that the relationship will be unprofitable going forward. Similarly, the principal may find it difficult to justify employing an agent known to be unprofitable - shareholders and board members may object.

Third, in the context of an organisation, we can think of the monitoring structure being something that applies to many agents at a particular level. In that case the monitoring structure is not something that is agent-specific, but organisation-wide. However, when deciding to fire an agent or not, the organisation has to deal with the individual and consider their value to the organisation. This decision is made on a case by case basis and is thus more flexible. [13]

[13] In an environment where monitoring decisions are publicly observable by many workers in an organisation, we could also imagine the commitment to monitoring arising as a larger equilibrium outcome. Say there are 100 employees at a level of the organisation, and the principal is meant to monitor each with probability half. This can be supported by a deviation to shirking by all workers, if they observe that the principal did not stick to this number.

4 Pure Adverse Selection: $u_H > 1$

First we consider the case in which $u_H > 1$. This corresponds to the situation in which the principal and the high type have perfectly aligned preferences and the high type strictly prefers effort to shirking. Under these preferences, the high type never shirks in any contract.[14] Therefore a contract in this environment only needs to incentivise the low type and the principal fires if and only if he observes the agent shirking. We can thus ignore the firing decision as a formal part of the model and write the game as in figure 2. A strategy profile

Low Type

Principal		E	S
	M	$-c, u_L$	$\frac{\delta\bar{w}}{1-\delta} - c, 0$
	N	$0, u_L$	$0, 1$

High Type

	E
M	$1-c$
N	1

Figure 1: The stage game

in this environment is $\sigma = (m, s_L)$ since the firing decision and the high type's strategy are given. Furthermore, since shirking being observed results in firing, relevant histories are reduced to sequences of N and E. In this section when I refer to the agent I mean the low type since the high type is non-strategic.

As a benchmark, consider the principal's best feasible outcome in the game, given a belief p. If the principal could observe the agent's type he would simply employ the high type forever without monitoring him and fire the low type immediately, giving the principal an ex-ante payoff of $p + (1-p)\bar{w}$. Given the constraint that he needs to monitor at least once to learn the agent's type, the highest feasible payoff through monitoring in the relationship is by **immediate full revelation**:

$$p + (1-p)\delta\bar{w} - c(1-\delta)$$

This involves having the high type exert effort, the low type shirk and monitoring with probability one today, after which the principal fires the low type, and keeps the high type forever with no more monitoring. If the low type shirks with probability less than one then the principal's payoff is strictly less than the best feasible outcome from monitoring. This

[14] Intuitively, the principal's limited commitment means that he can never credibly punish the high type for effort, so the high type always finds it dominant to exert effort. This is true under any reasonable refinement on off-path beliefs: if the principal observes shirking he should believe that the agent shirking is the one with the highest incentive to do so.

is obvious since the faster the principal learns, the earlier he can make an optimal firing decision, and as monitoring is costly, he prefers to learn everything at once rather than over time.

Define $\bar{p}$ as the belief at which the principal is indifferent between full revelation and employing the agent with no more monitoring. It satisfies

$$\bar{p} = \bar{p} + (1-\bar{p})\delta\bar{w} - c(1-\delta)$$

so

$$\bar{p} = 1 - \frac{c(1-\delta)}{\delta\bar{w}}$$

and $\bar{p} \in (0,1)$ as long as $\delta > \frac{c}{c+\bar{w}}$, that is if the principal is patient enough to find learning worthwhile. If the principal is too impatient to find experimentation of value then he will either never monitor and employ the agent forever, or not enter the relationship in the first place. For $p < \bar{p}$, the principal finds full revelation strictly optimal (conditional on entering the relationship).

4.1 Optimal Contract

In the pure adverse selection case, the definition of a contract can be reduced to:

Definition 2. *A contract is a profile σ such that s_L is incentive compatible for the agent at every history:*

$$V_L(\sigma)|_h \geq V_L(m, s'_L)|_h$$

for all s'_L, for all h.

An optimal contract is one which maximises the principal's expected payoff, $W(\sigma)$.

The main result is stated below. Note that it is stated for δ above a cutoff, but strictly less than 1. The qualitative content of this is that the players are patient enough that the principal finds costly experimentation worthwhile, the high type finds effort worthwhile if incentivised, and the low type finds effort worthwhile if rewards are high (and hence adverse selection is a problem).[15] To keep matters simple going forward, I will state all results

[15]If these conditions fail, then either the principal never bothers to monitor, or never enters the relationship, the high type never exerts effort, or the low type shirks no matter what. If any of these occur, we lose the interesting tension in the model.

where such conditions are needed in this manner. I emphasise that the exercise is not a folk theorem, which would consider the patient limit as δ goes to one.[16]

Theorem 1. *There exists $\delta^* \in (0,1)$ such that if $\delta > \delta^*$, there is a generically a unique optimal contract σ. There is a strictly decreasing sequence $\{m_k\}_{k=0}^n$ with $m_n = 0$ such that for all histories h,*
i) $m(h) = m_k$
where $k = n_h$ is the number of times monitored along h.
ii) $s_L(h) = 1$
The structure of the contract is in two phases:
Phase 1 (Screening) lasts until the first time the principal monitors.
Phase 2 begins if the principal observes effort the first time he monitors.

In the optimal contract, the principal's monitoring probability at any history depends only on the number of times he has monitored in the past. Phase 1 of the contract lasts while the principal has not monitored yet. In Phase 1, the principal monitors with a constant probability every period. As the low type shirks with probability one, the first time the principal monitors, he learns the agent's type, fires the agent if shirking is observed, and begins Phase 2 of the contract if effort is observed. Therefore the low type does not enter Phase 2 on path, and the principal enters Phase 2 with belief one that the agent is the high type. The monitoring probability is strictly decreasing in the number of times monitored in the past, and in Phase 2, the monitoring probability stochastically declines until eventually monitoring ends forever.

Notice that in the optimal contract, the principal continues to monitor *even after* he believes that the agent is the high type with probability one. This is clearly ex-post inefficient since the high type does not require incentives to exert effort. The purpose of this distortion is to make Phase 2 less attractive for the low type and be able to monitor with a higher probability in Phase 1. The more monitoring there is in Phase 2, the smaller the low type's expected payoff from entering Phase 2, and thus in Phase 1 he is willing to shirk when there is a higher risk of getting caught.

[16] In fact, whether a version of the folk theorem holds for this model or not depends on whether $u_H > 1$ or $u_H < 1$. In the former case, even in the patient limit, the principal cannot achieve first-best. In the latter the principal can achieve his first-best payoff of

$$p_0 + (1 - p_0)\bar{w}$$

which is what he could get if there was no adverse or moral hazard. This is only in the limit and the first-best is infeasible for every $\delta < 1$.

The optimal contract balances the costly distortion of monitoring in Phase 2 with the urgency of the principal's need to screen the low type. The principal picks the off-path value of the low type from Phase 2 to trade off these two things. The more pessimistic the principal is about the agent's type and the higher his outside option is, the faster he wants to screen and the lower value he will choose. Optimality of the contract requires that conditional on monitoring in Phase 2 and delivering some off-path value to the low type, the principal do it at the lowest monitoring cost possible.

To deliver some value less than 1 to the low type in Phase 2, the principal has to reduce the agent's payoff by monitoring enough. The optimal way to do this is to backload agent value. The principal can reduce the agent's value by making him willing to exert effort today and suffer the cost of effort. This requires monitoring today, and the higher is the agent's payoff after the principal observes effort, the smaller the monitoring probability needed. The intuition is that pushing agent payoffs into the future allows the principal to use lower powered incentives today. This results in the dynamics that arise: offering higher and higher values to the agent each time effort is observed means that the monitoring probability also declines each time. Outline of proof:

Step 1) Existence of Phase 1: I first show that there exists an optimal contract in which at every history where the principal has not monitored in the past, he monitors with a constant probability, has the agent shirk with probability one, and specifies the same off-path continuation value for the low type if he deviates to effort and is monitored (this is the low type's Phase 2 value). The higher this value is, the lower the initial monitoring probability needs to be.

Step 2) To understand the trade-off between the two phases, we need to solve for the principal's maximum payoff subject to delivering the low type any feasible v in Phase 2. The solution to this maximisation problem gives us a function $F(v)$ and I characterise the optimal monitoring policy for each v.

Step 3) The principal's overall payoff from a contract with the two phase structure can be written as a function of a single variable, v_L, the low type's time zero value from the contract, and is given by

$$W(v_L) = p_0 F(v_L) + (1 - p_0)(1 - v_L)\big(\delta\bar{w} - c(1 - \delta)\big)$$

Solving for an optimal contract then involves maximimising this function with respect to v_L, and generically, there exists a unique maximiser. This solves for an optimal contract, and generically this contract will be unique.

I now proceed with the proof. If σ is a contract, shirking is a best response for the agent at h if and only if

$$(1-\delta)u_L + \delta\big(1-m(h)\big)V_L(\sigma)|_{hN} + \delta m(h)V_L(\sigma)|_{hE} \leq \big(1-m(h)\big)\big((1-\delta)+\delta V_L(\sigma)|_{hN}\big)$$

or

$$m(h) \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta V_L(\sigma)|_{hE}}$$

I will use this inequality (and the opposite one) throughout the proof.

Existence of an optimal contract follows by a standard argument.[17]

4.2 Phase 1: Screening

Definition 3. *Let Phase 1 be the collection of histories h at which the principal has not monitored in the past:*

$$\textit{Phase } 1 := \{h^t \in \mathcal{H} | N \textit{ occurred in every period along } h^t\}$$

The proposition below shows that there exists an optimal contract of a particularly simple initial structure: At every history in Phase 1, the low type shirks with probability one and the principal monitors with a constant probability which is low enough to incentivise the agent to shirk. The contract specifies the same off-path continuation payoff for the agent after the first time principal observes effort. Thus Phase 1 of the contract is essentially stationary: conditional on not having monitored, the continuation contract looks identical to the one at time zero. This is optimal because the contract contingent on a history in Phase 1 does not affect incentives at any earlier history. Having the low type shirk while monitoring with positive probability is natural since it gives the principal the opportunity of catching the low type and firing him.

Proposition 1. *There exists $\delta^* \in (0,1)$ such that if $\delta > \delta^*$, there is an optimal contract σ such that for every h in Phase 1,*
i) $m(h) = m^* > 0$

[17]See appendix for proof.

ii) $s_L(h) = 1$

iii) $V_L(\sigma)|_{hE} = v$ *and*

$$m^* \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v}$$

Proof. First, note that if σ is an optimal contract, then conditional on no monitoring in the past, the continuation contract must be sequentially optimal: that is, the contract $\sigma|_h$ must be an optimal contract: for every h' on path to h, agent i's incentives to exert effort (shirk) are

$$m(h') \geq (\leq) \frac{(1-\delta)(1-u_L)}{1-\delta+\delta V_L(\sigma)|_{h'E}}$$

Notice that the incentives only depend on the contract at histories after $h'E$. Since h is not on path from $h'E$, the contract at h does not affect incentives at h'. Since σ is an optimal contract, $\sigma|_h$ must be an optimal contract too.

Second, let σ be an optimal contract and $w = W(\sigma)$. We will show that for δ sufficiently high, at every history h in Phase 1, it must be that $s_L(h) = 1$ and

$$0 < m(h) \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta V_L(\sigma)|_{hE}}$$

By sequential optimality continuation contracts after no monitoring in the past, it is sufficient to show the claim for $h = \emptyset$. Let $p_0 < \bar{p}$, so the principal finds full revelation strictly optimal compared to never monitoring, and $\delta > \frac{c}{c+\bar{w}(1-p_0)}$. Then $w > p_0$: consider the contract which monitors with probability ϵ at time zero and then stops monitoring forever afterwards. For ϵ sufficiently small, the low type strictly prefers to shirk at time zero, so the principal's payoff from the contract is

$$\epsilon[p_0 + (1-p_0)\delta\bar{w} - c(1-\delta)] + (1-\epsilon)p_0 > p_0$$

where the strict inequality follows if $\delta > \frac{c}{c+\bar{w}(1-p_0)}$.

Suppose, for the sake of contradiction, that $m(h) = 0$. Then it must be that

$$w = (1-\delta)p_0 + \delta w$$

a contradiction since $p_0 < w$. Therefore $m(h) > 0$.

Suppose $s_L(h) = 0$. Then

$$w = (1-\delta)\big(p_0 - m(h)c\big) + \delta\big(1-m(h)\big)w + \delta m(h)W(\sigma)|_{hE}$$

Now $p_0 - m(h)c < w$ since $w > p_0$. This implies that $W(\sigma)|_{hE} > w$. Since $p(hE) = p_0$, the contract $\sigma|_{hE}$ is feasible at time 0, contradicting that σ is an optimal contract.
Suppose $s_L(h) \in (0, 1)$. Then shirking is incentive compatible for the agent. Then clearly, the contract which is identical to σ everywhere apart from $s_L(h) = 1$ is incentive compatible and a strict improvement for the principal, a contradiction. Therefore $s_L(h) = 1$, and the agent's incentives require that $m(h) \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta V_L(\sigma)|_{hE}}$.
Now construct the contract σ' as the contract that specifies σ at every history in Phase 1. Then clearly σ' satisfies i), ii) and iii). Furthermore, by 1),

$$w = \frac{(1-\delta)\big(p_0 - m(\emptyset)c\big) + \delta m(\emptyset)\big(p_0 W(\sigma)|_{\emptyset E} + (1-p_0)\bar{w}\big)}{1-\delta\big(1-m(\emptyset)\big)} = W(\sigma')$$

so σ' is an optimal contract. □

Thus to find an optimal contract we can focus on a particularly simple form of contract. These are contracts in which at every history in Phase 1, the agent shirks and the principal monitors with the same probability, and specifies the same continuation contract after monitoring and observing effort. I will call these **simple contracts**.

Let σ be an optimal contract that is simple. Then at every history in Phase 1, the agent's incentive constraint requires that

$$m^* \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v}$$

The principal eventually monitors after some history. If the agent is found to be shirking, the principal fires him as he is revealed to be the low type. If he is exerting effort, the principal begins Phase 2 of the contract in which he knows that the agent is the high type, with the constraint that he has to deliver the promised off-path value v to the low type. v is thus the maximum value that the low type can get if he deviates to effort and is observed by the principal, convincing the principal that he is the high type.

Notice that the lower v is, the larger the upper bound on m^* is. This is good for screening since increasing m^* results a shorter expected time until the principal monitors in Phase 1. The trade-off that the principal faces is that of choosing faster screening versus the cost of delivering lower values in the future. Consider the extreme cases:

i) No monitoring in Phase 2, $v = 1$: clearly the ex-post efficient thing to do once faced with a known high type is to stop monitoring since the high type never shirks. Then the low type,

conditional on deviating and entering Phase 2, can shirk forever without being monitored. In this case the Phase 1 monitoring probability is at most $m^* = (1-\delta)(1-u_L)$, resulting in relatively slow screening.
ii) Monitor forever with probability 1 in Phase 2, $v = 0$. In this case, the principal can monitor with probability one in Phase 1 and screen immediately, but the cost of monitoring in Phase 2 is very high.

An optimal contract therefore trades off the distortions to the contract in Phase 2 with the need to screen the low type quickly. In order to understand this trade-off, we need to solve for the maximal payoff of the principal conditional on delivering a value v in Phase 2.

4.3 Phase 2: Optimal Delivery of a value v

Having entered Phase 2 the principal has to deliver the promised off-path value for the low type, v. We can think of this as a time zero contract σ with initial belief $p_0 = 1$ and a constraint that $V_L(\sigma) = v$. The principal's problem is then to maximise his payoff subject to delivering v:

$$\begin{aligned} F(v) := \max_{\sigma} \ & W(\sigma) \\ \text{subject to} \quad & V_L(\sigma) = v && (PK) \\ & V_L(\sigma)|_h \geq V_L(s'_L, m) \ \forall s'_L \ \forall h && (IC) \end{aligned}$$

Notice that the (low type) agent behaviour does not enter the principal's payoff since the low type is not there on-path. We can assume without loss that the agent takes only pure actions. The promise-keeping constraint (PK) and incentive compatibility still have to hold in order to credibly deliver the off-path value v to the agent. $F(v)$ denotes the solution to this problem.

I formulate the problem recursively in the manner of Abreu, Pearce and Stacchetti (1990) (APS henceforth). Define the set of feasible pairs of payoffs for the principal and the agent as

$$\mathcal{E} := \{ \big(W(\sigma), V_L(\sigma)\big) | \ \sigma \text{ a contract}, \ p_0 = 1 \}$$

A pair of values in $\mathcal{E}$ is then generated by a pair of actions today: a monitoring probability for the principal and a shirking decision by the agent, and continuation values the principal and the agent after each public event, N and E, which must come from $\mathcal{E}$ themselves.

Definition 4 (Contract values)**.** *Let $(w, v) \in \mathcal{E}_H$. Then there exist actions $m \in [0, 1]$, $s_L \in \{0, 1\}$, and continuation values $(w^N, v^N), (w^E, v^E) \in \mathcal{E}_H$ such that promise-keeping for each player, and the agent's incentive constraint holds:*

$$w = (1-\delta)(1-mc) + \delta(1-m)w^N + \delta m w^E \tag{PPK}$$

$$v = (1-\delta)\big[(1-s_L)\, u_L + s_L\,(1-m)\big] + \delta(1-m)v^N + \delta m(1-s_L)v^E \tag{APK}$$

$$s_L \in \arg\max_{s' \in \{0,1\}} \; (1-\delta)\big[(1-s')u_L + s'(1-m)\big] + \delta(1-m)v^N + \delta m(1-s')v^E \tag{AIC}$$

By APS, $\mathcal{E}$ is compact. Note that every $v \in [0, 1]$ can be feasibly delivered to the agent with the policy $m(h) = \frac{(1-\delta)(1-v)}{1-\delta+\delta v}$, and $s_L(h) = 1$ for all h. Therefore $F : [0, 1] \to [0, 1]$ and

$$F(v) = \max\{w | (w, v) \in \mathcal{E}\}$$

F is then the fixed point of the operator $T : \mathcal{B}[0, 1] \to \mathcal{B}[0, 1]$[18]

$$Tf(v) = \max_{\substack{m\in[0,1], s_L\in\{0,1\},\\ v^N, v^E\in[0,1]}} (1-\delta)(1-mc) + \delta(1-m)f(v^N) + \delta m f(v^E)$$

$$\text{subject to} \quad v = (1-\delta)(1-m)s_L + \delta\left[(1-m)v^N + m(1-s_L)v^E\right] \tag{PK}$$

$$s_L \in \arg\max_{\tilde{s}} \; (1-\delta)(1-m)\tilde{s} + \delta\left[(1-m)v^N + m(1-\tilde{s})v^E\right] \tag{IC}$$

A standard check of Blackwell's sufficient conditions verifies that T is a contraction on the space $\mathcal{B}[0, 1]$ and therefore has a unique fixed point. The following proposition fully characterises the solution to the optimal delivery problem.

Proposition 2. *There exists a strictly decreasing sequence $\{v_k\}_{k=0}^K$, with $v_0 = 1$ and $v_K = 0$, such that*

$$F(v) = 1 - c(1-\delta)\sum_{i=0}^{k-1} \delta^i (1-u_L)^i \left(1 + (k-1-i)u_L\right) + c(1-\delta)\sum_{i=0}^{k-1}(k-i)v$$

for $v \in [v_k, v_{k-1})$, and F is strictly increasing, piecewise linear and concave.
For $v \in [v_k, v_{k-1})$, the optimal policy must satisfy:
i) $v^N \in [v_k, v_{k-1}]$, $v^E \in [v_{k-1}, v_{k-2}]$.
ii) If $v \leq v_1$, then IC binds.

[18] $\mathcal{B}[0, 1]$ is the space of bounded functions on $[0, 1]$.

iii) If $v = v_k$ for $k \in \{1, ..., K-1\}$ then $v^N = v$, $v^E = v_{k-1}$ and

$$m = \frac{(1-\delta)(1-v)}{1-\delta+\delta v}$$

Proof. See Appendix. □

To understand the structure of F and the optimality of the policies stated in the proposition, it helps to think about a simple and intuitive policy that we can conjecture is optimal. It then turns out that using these policies we can construct F explicitly and verify its optimality using the operator T.

First, $F(1) = 1$ as the principal never needs to monitor to deliver 1. An obvious guess is that F will be strictly increasing. After all, delivering higher values ought to require less monitoring and therefore should be less costly. Let v be the value to be delivered. If F is strictly increasing then since the agent's incentives depend only on the continuation value v^E, it must be that either $v^E = 1$ or the IC binds. Otherwise we could raise v^E without affecting incentives and improve the principal's payoff in the event that E occurs. Furthermore, if the agent is exerting effort, then IC ought to bind. Having the agent exert effort is good for reducing the agent since the agent receives u_L today, but if the IC is slack, the principal can lower the monitoring probability and reduce his cost today while maintaining incentives. As a result it is without loss to always have the agent shirk since his action does not enter the principal's payoff. Finally, if N occurs today the principal observes and has nothing to condition on to punish or reward the agent. Therefore an intuitive guess would be that setting $v^N = v$ is optimal. These conjectures reduce PK to

$$v = (1-m)(1-\delta+\delta v)$$

and IC to

$$m \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E}$$

With this in hand, we can begin to construct F recursively. If v is very close to 1 then it must be that the IC is slack, since giving the agent a payoff of u_L today will make PK infeasible. Then by our conjecture, $v^E = 1$ and the principal's payoff is

$$F(v) = \frac{(1-\delta)(1-mc)+\delta m}{1-\delta(1-m)}$$

where m satisfies PK and we use the guess that $v^N = v$. This gives us the first piece of the function

$$F(v) = 1 - c(1-\delta)(1-v)$$

This can be generated by setting $v^E = 1$ as long as the m implied from PK satisfies IC:

$$m = \frac{(1-\delta)(1-v)}{1-\delta+\delta v} \leq (1-\delta)(1-u_L)$$

The lowest v for which it is feasible to have $v^N = v$, $v^E = 1$ and satisfy PK and IC is v_1, which satisfies

$$v_1 = (1-m)(1-\delta+\delta v_1)$$

and

$$m = (1-\delta)(1-u_L)$$

so the IC binds. Below v_1 we need to start lowering v^E from 1, which means that the IC will bind by our conjecture. Then if $v^N = v$, IC pins down v^E. For v just below v_1, this results in $v^E \in [v_1, 1]$, where we have a guess for the function. Therefore we can construct the next piece of F as

$$F(v) = \frac{(1-\delta)(1-mc) + \delta m \left(1 - c(1-\delta)(1-v^E)\right)}{1-\delta(1-m)}$$

This construction is valid as long as setting $v^N = v$ and having PK and IC bind results in $v^E \in [v_1, 1]$, so as long as $v \in [v_2, v_1]$, where v_2 satisfies

$$v_2 = (1-m)(1-\delta+\delta v_2)$$

and

$$m = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^1}$$

We continue constructing F piecewise in this way, and construct the sequence $\{v_k\}_{k=0}^{K}$. which satisfies the recursion $v_0 = 1$,

$$v_k = (1-m)(1-\delta+\delta v_k)$$

and

$$m = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v_{k-1}}$$

for $k \in 1, ..., K-1$. The sequence converges to something strictly less than 0, so we defined $K-1$ as the lowest positive point in the sequence, and set $v_K = 0$. By setting $v^N = v$,

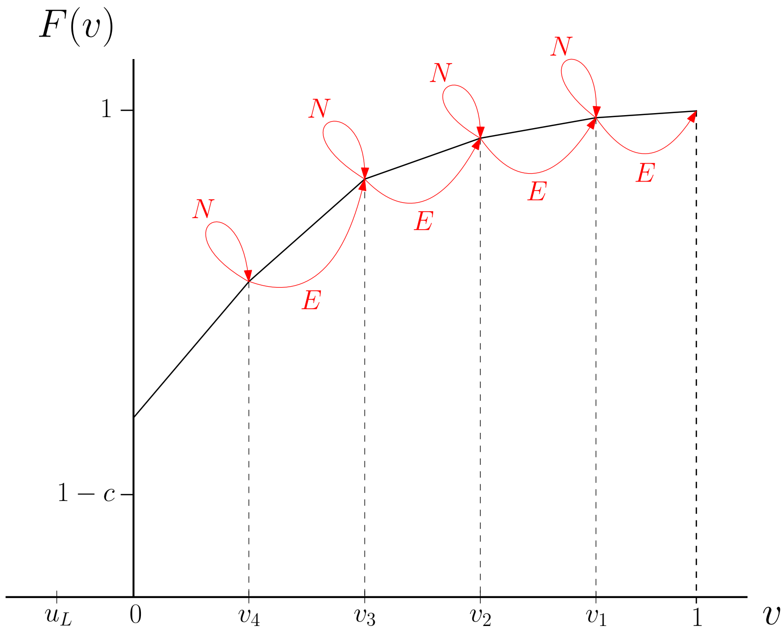


Figure 2: F and optimal policy dynamics.

PK and IC binding imply that if $v \in [v_k, v_{k-1}]$ then v^E lies in $[v_{k-1}, v_{k-2}]$. Therefore the continuation values are always in adjacent intervals. Given an interval above, v_k is the lowest v such that we can still set $v^N = v$, satisfy PK and IC binding and set v^E in the adjacent interval above. The kinks of F occur at the points of the sequence. Once we have constructed the function, the proof verifies that it is indeed the fixed point of the operator T to show that it is optimal. The proof also highlights that for values in the interior of the interval (v_k, v_{k-1}), there are multiple optimal policies (although the one used in the construction is the simplest): any policy such that the v^N is in the same interval as v and v^E in the adjacent interval above with IC binding and PK holding is indeed optimal. However, if we begin on one of the kinks, v_k, then the policy is uniquely pinned down. since we can only satisfy the interval requirement setting v^N and v^E on the kinks, and then the same applies tomorrow since we remain on the kinks, and eventually the value hits 1.

The movement of the continuation values upwards after each time the principal monitors and observes effort means that the monitoring probability is declining over time. If we focus on the policy in which $v^N = v$, then PK means that

$$m = \frac{(1-\delta)(1-v)}{1-\delta+\delta v}$$

and $v^E > v$, so the monitoring probability will decrease each time E is observed. The upward movement of agent continuation values and decreasing of monitoring probability is the backloading of agent payoffs. This property appears in many dynamic contracting settings and is an extremely robust finding (see Lazear (1981), Harris and Holmström (1982), Ray (2002)). The idea is that since the principal is going to have to monitor, he should get the most bang for his buck - the best way to do this is to make the agent willing to exert effort and suffer a payoff of u_L today. Furthermore, the more value the agent gets after the principal observes effort, the lower the monitoring probability needed to make the agent indifferent. Therefore pushing agent value into the future allows the principal to lower his monitoring costs. In this particular problem, the principal's rewards are also being backloaded: the principal's payoff is increasing in the agent's value, and the principal's costs - monitoring - are frontloaded.

4.4 Contract Optimisation

I now return to the optimal choice of policy at the screening stage of the contract. Let σ be an optimal contract that is simple. The following proposition shows that in Phase 1, the agent's incentive constraint to shirk must bind. This is intuitive: if the incentive constraint is slack and the monitoring probability is less than one, the principal can increase the monitoring probability to improve his payoff by learning faster, while maintaining the agent's incentive to shirk. If the monitoring probability is one, then the principal can increase the promised value v for an improvement since $F(v)$ is strictly increasing in v.

As the agent is shirking and his incentive constraint binds, choosing v is equivalent to choosing $v_L := V_L(\sigma)$. In any such contract, the agent's value satisfies

$$v_L = (1 - m^*)(1 - \delta + \delta v_L)$$

and the Phase 1 incentive constraint is

$$m^* = \frac{(1 - \delta)(1 - u_L)}{1 - \delta + \delta v}$$

This allows us to characterise the principal's payoff from an optimal contract that is simple in terms of a single variable, the low type's time zero value from the contract or equivalently, the low type's Phase 2 value. We get a nicer functional form if we use the time zero value v_L, so I write it in this way.

Proposition 3. *There exists $\delta^* \in (0,1)$ such that for all $\delta > \delta^*$, there is an optimal contract σ that is simple such that at every h in Phase 1, the agent's incentive constraint to shirk binds:*

$$m^* = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v}$$

and the principal's payoff from the contract is

$$W(\sigma) = p_0 F(v_L) + (1-p_0)(1-v_L)\big(\delta\bar{w} - c(1-\delta)\big) =: W(v_L)$$

where $v_L := V_L(\sigma)$.

Proof. See Appendix. □

The function $W(v_L)$ generically has a unique maximiser at one of the kinks of the F function. As a result, there is generically a unique optimal contract, which is a simple contract.

Proposition 4. *There exists $\delta^* \in (0,1)$ such that if $\delta > \delta^*$, there is a generically a unique optimal contract σ. It delivers low type value $V_L(\sigma) = v_{n^*}$, for some $n^* \in \{2, ..., K\}$, which is the generically unique maximiser of $W(v_L)$, and has Phase 1 monitoring probability*

$$m^* = \frac{(1-\delta)(1-v_{n^*})}{1-\delta+\delta v_{n^*}}$$

and optimally delivers Phase 2 value v_{n^*-1}.

Proof. See Appendix. □

Theorem 1 is an immediate consequence of this result. By proposition 2, the optimal policy in Phase 2 is uniquely pinned down since Phase 2 begins with delivering v_{n^*-1}, and continuation values remain on the kinks thereafter.

4.5 Equilibrium with No Commitment

The principal's commitment power to monitoring is strongly binding in the optimal contract. A natural question to ask is what if the principal has no commitment? I focus on public perfect Bayesian equilibria (PPBE) of the game. The principal does not have commitment power so an equilibrium is a profile σ such that at every history h, each player is playing a best response.

It turns out that the monitoring technology in this environment has fairly limited value. In particular, monitoring has value only if the principal is sufficiently pessimistic about the agent's quality initially. The main result is:

Theorem 2. *There exists $n^* \geq 2$ such that generically*[19]*:*
i) If $p_0 > \overline{p}^{n^*+1}$ there exists a unique equilibrium (σ) which is Markovian with respect to the belief p. Equilibrium payoffs are

$$W(\sigma) = p_0$$

and

$$V(\sigma) = \frac{\delta^n(1-u_L)^n + u_L \sum_{i=0}^{n-1} \delta^i (1-u_L)^i}{\sum_{i=0}^{n} \delta^i (1-u_L)^i}$$

where n is such that $p_0 \in (\overline{p}^{n+1}, \overline{p}^n)$.
ii) If $p_0 < \overline{p}^{n^*+1}$ then equilibrium exists, with the agent's strategy s_L uniquely defined and Markovian with respect to the belief p. If (σ) is an equilibrium, payoffs are uniquely given by

$$W(\sigma) = p_0 + \left(1 - \frac{p_0}{\overline{p}^{n^*}}\right) \delta \overline{w} - c(1-\delta)$$

and

$$V(\sigma) = 0$$

Proof. See Appendix. □

Generically, equilibrium payoffs are unique. The theorem states that if the prior is sufficiently high, there are no gains from monitoring for the principal. Furthermore, the equilibrium is generically unique and Markovian in p. It exhibits strict mixing by both players until the belief crosses $\overline{p}$ (the belief at which the principal is indifferent between learning the agent's type and employing him without ever monitoring again) or the agent is fired. If the prior is low enough then there are gains from monitoring. The intuition for the result is as follows. In order for there to be gains from monitoring, it must be that principal strictly prefers to monitor at some point while the agent shirks with positive probability. This implies that the agent's payoff at that history is 0. Recall that once the belief crosses $\overline{p}$, the principal will stop monitoring forever. If the belief is sufficiently close to $\overline{p}$ then the agent can guarantee himself a strictly positive payoff by exerting effort to deceive the principal until the belief

[19] The result is generic in that it holds for all but a Lebesgue measure zero set of priors p_0. The non-generic case arises if $p_0 = \overline{p}^n$ for some n, in which case the agent's equilibrium payoff may be non-unique.

crosses $\overline{p}$. Therefore the principal cannot deliver the agent a payoff of 0 and there are no gains from monitoring. If the belief is low, however, the expected amount of time exerting effort it takes to deceive the principal is too high and cannot guarantee the agent a positive payoff. Thus the principal is able to deliver the agent a payoff of 0 and gains from monitoring exist.

Since the agent plays a unique strategy in equilibrium, and this strategy is Markovian with respect to the belief, the equilibrium path of the belief conditional on the agent not being fired is a uniquely defined sequence. In fact, in equilibrium, after a finite number of periods in which the principal monitors and does not catch the agent, the belief crosses $\overline{p}$ and the belief stops moving. Therefore this path is finite.

Proposition 5. *There exists a unique finite sequence, $p_0 <, ..., < p_N$, such that in any equilibrium, conditional on the agent not being fired, the principal's belief follows this path. It satisfies*

$$p_{i+1} = \frac{p_i}{\overline{p}}$$

for all $i \geq 1$, and

$$p_1 = \begin{cases} \frac{p_0}{\overline{p}} & \text{if } p_0 > \overline{p}^{n^*+1} \\ \overline{p}^{n^*} & \text{if } p_0 < \overline{p}^{n^*+1} \end{cases}$$

Proof. See Appendix. □

Given an equilibrium action s_L today by the agent, the principal's updated belief after monitoring and observing effort is

$$p' = \frac{p}{1 - s_L(1-p)}$$

so there is a one-to-one map between the path of beliefs and the agent's strategy. Therefore we can think of the agent as choosing the path of the belief. In equilibrium the belief increases a finite number of times until either the agent is caught or the belief crosses $\bar{p}$, after which the principal stops monitoring. Apart from at time 0, whenever $p < \overline{p}$, the agent's strategy is defined by

$$p' = \frac{p}{\overline{p}}$$

This makes the principal indifferent between monitoring and not monitoring. Therefore we have strict mixing by both players until the belief crosses $\overline{p}$. This is naturally the kind of equilibrium we would expect in a game like this. When $p_0 < \overline{p}^{n^*+1}$, playing the same strategy as above does indeed make the principal indifferent between monitoring and not. However, playing this strategy from such a low prior requires the agent to be willing to exert effort for too long, giving him a negative expected payoff. Therefore at time zero, a larger jump in the belief is needed to take it to a region in which a mixed equilibrium is feasible. For such a large jump in the belief, monitoring is a strict best response for the principal. The principal monitors for sure while the agent shirks with some probability. If the agent is not caught, the belief is updated and the fully mixed equilibrium begins from the following period. Therefore after time zero, the principal weakly prefers to not monitor. In this sense, all of the gains from monitoring come in the very first period of the relationship. This is the content of the next result.

Proposition 6. *Let (σ) be an equilibrium, $t \geq 1$ and h^t an on-path history. Then*

$$W(\sigma)|_{h^t} = p(h^t)$$

Proof. See Appendix. □

This is a common property of equilibria of timing games and wars of attrition in which one of the players concedes with positive probability at time zero, after which play moves to a balanced path in which both players are mixing and evenly matched in strength.[20] The structure of the equilibrium is similar to those which arise in models of reputation in repeated games.

In relationships with reputation building, it is common that as an employer's trust in his employee grows, the employee is left more and more to his own devices. The more reliable the employer believes the employee to be, the less he monitors. I find that this is indeed what happens in equilibrium. On the equilibrium path, as the principal's belief increases, his monitoring probability decreases, until eventually he stops monitoring forever. Formally, given the path of beliefs in equilibrium, the monitoring probability at a higher belief is weakly lower than the monitoring probability at a lower belief. This is strict if either the beliefs are two jumps apart or the principal's strategy is Markovian.[21]

[20] See for instance Abreu and Gul (2000), Hendricks, Weiss and Wilson (1988)

[21] It is possible to construct an equilibrium in which there are two beliefs, p and p' on the equilibrium path such that there is a history h with $p(h) = p$, and a history h' with $p(h') = p'$ with $m(h) = m(h')$. The simplest one is with $p = \overline{p}$ and $p' = 1$. Set $m(h) = 0$, $m(hN) > 0$ and $m(hNE) = 0$. Examples below $\overline{p}$ exist and require a similar kind of non-stationary construction in which the monitoring probability at p is the

Proposition 7. *Let* (σ) *be an equilibrium and let* $p_0, .., p_N$ *be the unique path of the equilibrium beliefs. If* h *and* h' *are histories such that* $p(h) = p_i$ *and* $p(h') = p_j$*, with* $0 \leq i < j \leq N$*, then*

$$m(h) \geq m(h')$$

and the inequality is strict if $j > i + 1$ *or* m *is Markovian with respect to* p*.*
For h *such that* $p(h) = p_N$*,* $m(h) = 0$*.*

Proof. See the Appendix. □

5 Moral Hazard and Adverse Selection: $u_H < 1$

I now introduce the additional strategic behaviour of the high type of agent. If $u_H < 1$ then the high type strictly prefers shirking and getting away with it to exerting effort. Therefore the principal faces a problem of moral hazard as well as the adverse selection due to the low type. The high type now needs to be incentivised to exert effort.

I restrict attention to contracts that specify pure strategies for the agent. $s_H, s_L : \mathcal{H} \to \{0, 1\}$ so each type of agent makes a binary decision to shirk or exert effort at every history. While this is without loss of generality in the pure adverse selection case, I do not know if it is without loss for this case. In theory, due to the principal's inability to continue a relationship with an agent known to be the low type, he may benefit from inducing intermediate beliefs by having the high type mix, thus allowing him to continue the relationship after seeing shirking and becoming more pessimistic.

5.1 Optimal Contract

Theorem 3. *There exists* $\delta^* \in (0, 1)$ *such that if* $\delta > \delta^*$*, there is an optimal contract* σ *with the following structure. There exists a strictly decreasing sequence* $\{m_k\}_{k=0}^{n^*}$ *and* $T \in \mathbb{N}$ *such that for any history* h*,*

i) $s_L(h) = 1$.

ii) if S *has not occurred in the past,*

highest possible at h, and the lowest possible at hN. Apart from such constructions, monitoring is strictly decreasing in the belief.

$$
\begin{aligned}
d(h) &= 1 \\
m(h) &= \begin{cases} m_{n_h} & \text{if } n_h < n^* \\ 0 & \text{if } n_h \geq n^* \text{ and } E \text{ occurred within the last } T \text{ periods} \\ m_{n^*} & \text{if } n_h \geq n^* \text{ and } N \text{ occurred in each of the last } T \text{ periods} \end{cases} \\
s_H(h) &= \begin{cases} 0 & \text{if } n_h < n^* \\ 1 & \text{if } n_h \geq n^* \text{ and } E \text{ occurred within the last } T \text{ periods} \\ 0 & \text{if } n_h \geq n^* \text{ and } N \text{ occurred in each of the last } T \text{ periods} \end{cases}
\end{aligned}
$$

iii) if S has occurred in the past,

$$
\begin{aligned}
d(h) &= 0 \\
m(h) &= 0 \\
s_H(h) &= 1
\end{aligned}
$$

Phase 1 of the contract lasts until the first time the principal monitors.
Phase 2 begins if the principal observes effort the first time he monitors.

Structure of the contract:

- **Phase 1 (Screening)**: As long as the principal has not actually monitored in the past, the principal continues the relationship and monitors with a constant probability every period, the high type exerts effort and the low type shirks. The first time the principal monitors, he observes the agent's action. If the agent is shirking the principal fires the agent next period. If the agent is exerting effort, the principal begins Phase 2 of the contract with belief one that the agent is the high type.

- **Phase 2 (Relationship with high type)**: The principal never fires the agent on path. The principal begins with a monitoring probability strictly less than that in Phase 1. Each time the principal actually monitors, the monitoring probability decreases. The high type exerts effort as long as the principal is monitoring with positive probability. After the principal has monitored a fixed number of times, $n^* - 1$, the relationship enters a cycle which continues forever: The principal stops monitoring for T periods, while the high type shirks. After T periods, the principal monitors with a constant probability which is the lowest used in the contract, and the high type exerts effort, until the principal actually monitors.

If the principal ever observes shirking the high type begins shirking forever and the principal immediately fires in the next period. The low type always shirks (off-path).

Phase 1 of the contract is identical to that in the pure adverse selection case. The difference arises in Phase 2 after the principal has learned that the agent is the high type. The principal wants to incentivise the high type to work using monitoring optimally. The best way to do this is to backload high type value, providing high rewards after observing effort. This reduces monitoring costs because if there are high rewards for effort, lower monitoring probabilities incentivise effort. The only way to reward the high type is to allow him to shirk after some histories. This is costly for the principal, so the rewards are backloaded and shirking is allowed after a sufficiently long time working. This allows the most effective use of the rewards. Every time the principal monitors, the reward stage comes closer and therefore the principal can offer higher values after observing effort. This is why the monitoring probability decreases after each time the principal monitors and the contract gradually moves towards the stage where the high type is allowed to shirk for some time as a reward. Due to the perfect monitoring technology, there is no firing on path. The high type is never monitored shirking and the harshest off-path punishment of firing is provided if the principal ever observes shirking.

The trade-off identified in the adverse selection case remains: the more the principal monitors in Phase 2, the faster he can screen the low type in Phase 1. An analogous but more subtle distortion arises in this case. Phase 2 of the relationship begins inefficiently in order to reduce the low type's off-path value from deviating. The inefficiency that arises is that the principal initially monitors with higher probability than is necessary to make the high type exert effort, and the high type works for longer than is efficient. Formally, the relationship between the principal and the high type in Phase 2 begins strictly off the Pareto frontier of the relationship, a distortion that arises due to the adverse selection problem. After the principal has monitored a given number of times, the continuation contract hits the Pareto frontier at the principal-optimal point. Once the contract hits the Pareto frontier, it stays on there, converging to the agent-best point over time. This is stated in the following result which is a corollary of the main theorem.

Corollary 1. *There exists $\delta^* \in (0,1)$ such that if $\delta > \delta^*$, there exists $M \in \mathbb{N}$ such that Phase 2 of the optimal contract is inefficient until the principal has monitored M times.*

The proof of the main theorem is structured in the same way as that for the case of pure adverse selection. The key difference and greater difficulty lies in the construction of the solution for Phase 2.

The analogous result for Phase 1 is:

Proposition 8. *There exists $\delta^* \in (0,1)$ such that if $\delta > \delta^*$, there exists an optimal contract σ such that for every h in Phase 1,*
i) $m(h) = m^* > 0$
ii) $s_H(h) = 0$, $s_L(h) = 1$
iii) $V_L(\sigma)|_{hE} = v$ and

$$\frac{(1-\delta)(1-u_H)}{1-\delta+\delta V_H(\sigma)|_{hE}} \leq m^* \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v}$$

Proof. See Appendix. □

Again we will call a contract with the above properties a **simple contract**. As before, once the principal monitors and observes effort, he must deliver the off-path value v in Phase 2. The principal faces the trade-off when choosing the value v: faster screening compared to the amount of monitoring required if Phase 2 begins.

5.2 Optimal Delivery of a Value v

In Phase 2, the principal must credibly deliver the promised off-path value v to the low type. At the same time, he must incentivise the high type to work. Therefore, the principal's problem is to choose a contract to maximise his payoff in the relationship with the high type subject to the constraint that a low type faced with that contract should get a value of v. We can think of this as a time zero contract with initial belief $p_0 = 1$ and the constraints that $V_L(\sigma) = v$, incentives hold for each type at every history (ICH and ICL), and the principal's firing decision is optimal at every history (PFD). The principal's problem, for any v is:

$$\begin{aligned} F(v) := \max_{\sigma} \;& W(\sigma) \\ \text{subject to} \quad & V_L(\sigma) = v && (PK) \\ & \textit{ICH, ICL } \forall h && (IC) \\ & \textit{PFD } \forall h \end{aligned}$$

I recursively formulate the problem. Formally, due to the high type's incentives, we need keep track of the high type's continuation values, but it turns out that we can fully characterise the solution in terms of continuation values for the low type. For the formal recursive formulation of the problem, see the appendix. If the principal is delivering the low type a value v today, he needs to specify actions today (continuation probability d, monitoring probability m, and shirking decisions s_H and s_L) and continuation values v^N, v^E, and v^S for the low type after the different public events. The reason that it is sufficient to only specify low type continuation values is that due to the single-crossing condition on the payoffs from effort, the high type always receives weakly higher values than the low type, and only one type's incentives can bind. We can show that whenever the high type's incentive constraint binds, both types receive the same value and therefore the low type's values are sufficient to check the high type's incentives. The proposition below fully characterises the solution to the problem:

Proposition 9. *There exists $\delta^* \in (0,1)$ such that if $\delta > \delta^*$: there exists $1 > \bar{v} > u_H$ such that $F : [0, \bar{v}] \to [0,1]$ and there is a strictly decreasing sequence $\{v_k\}_{k=0}^{K}$ with $v_0 = \bar{v}$ and $v_K = 0$ and $n \in \{1, ..., K-1\}$ such that*

$$F(v) = A_k + B_k v$$

for $v \in [v_k, v_{k-1})$ for some A_k, B_k, and F is piecewise linear, concave, strictly increasing for $v \geq v_n$ and strictly decreasing for $v < v_n$. where $v_n \in (u_H, \bar{v})$.
For $v \in [v_k, v_{k-1})$, the optimal policy can be characterised in terms of the low type's value today and continuation values, v^N, v^E, v^S and satisfies:
i) $d = 1$ ii) $s_L = 1$ and $s_H = \begin{cases} 1 & \text{if } v > (1-\delta)u_H + \delta\bar{v} \\ 0 & \text{if } v \leq (1-\delta)u_H + \delta\bar{v} \end{cases}$

iii) $m = \begin{cases} 0 & \text{if } v > (1-\delta)u_H + \delta\bar{v} \\ \frac{(1-\delta)(1-u_H)}{1-\delta+\delta\bar{v}} & \text{if } v \in [v_1, \bar{v}] \\ \frac{(1-\delta)(1-v)}{1-\delta+\delta v} & \text{if } v < v_1 \end{cases}$

iv) $v^S = 0$, $v^N \in [v_k, v_{k-1}]$ *and* $v^E \in \begin{cases} [v_{k-1}, v_{k-2}] & \text{if } k \leq n+1 \\ [v_{k-2}, v_{k-3}] & \text{if } k > n+1 \end{cases}$

v) If $v \geq v_{n+1}$ then ICH binds.
If $v \leq v_{n+2}$ then ICL binds.

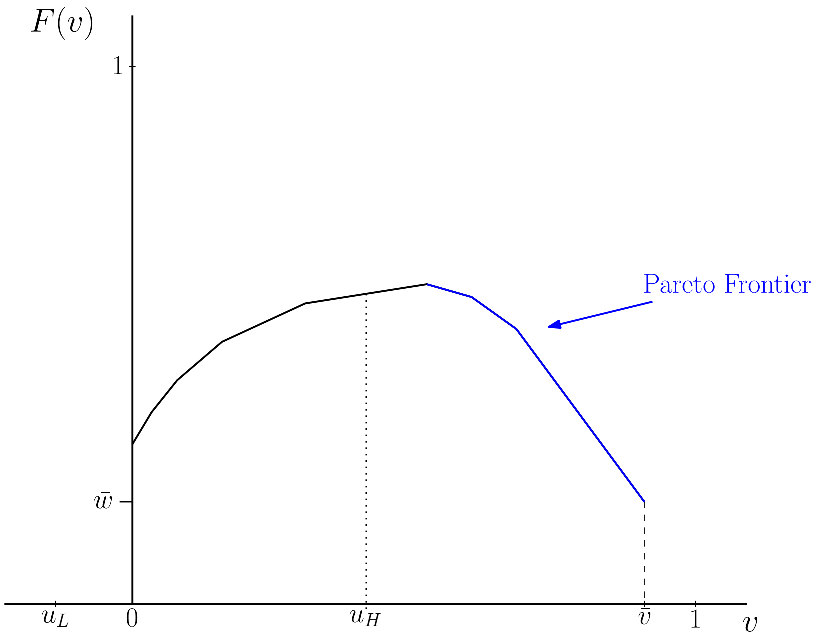


Figure 3: Optimal delivery function F.

vi) If $v = v_k$ for some k then $v^N = v$ and $v^E = \begin{cases} v_{k-1} & \text{if } k \leq n+1 \\ v_{k-3} & \text{if } k > n+1 \end{cases}$

Proof. See Appendix. □

F is a piecewise linear and concave function, and the kinks of F occur at a sequence of points $\{v_k\}_{k=0}^{K}$. The function is maximised at one of the kinks, $v_n \in (u_H, \bar{v})$, is strictly increasing below, and strictly increasing above. In the optimal policy, the low type never strictly prefers to exert effort since then the principal can lower costs by reducing the monitoring probability. Therefore the low type always shirks without loss of generality since his actions do not enter the principal's payoffs (on-path). It is optimal to have the high type exert effort whenever feasible: as long as $v \leq (1-\delta)u_H + \delta\bar{v}$. To deliver higher values than this to the low type requires that punishments for shirking are too low to incentivise the high type to work, so the high type must be allowed to shirk. When the high type shirks, there is no monitoring as there is nothing to incentivise. Therefore there is no firing or shirking on path: if the principal ever observes shirking he fires the agent (enforced by the high type shirking forever if shirking occurs on path), so $v^S = 0$ at any value.

To deliver $v \in [v_n, (1-\delta)u_H + \delta\bar{v}]$, the values are sufficiently high that it is possible to monitor efficiently: the high type's incentive constraint to exert effort binds. In any optimal policy, for $v \in [v_k, v_{k-1})$, it must be that v^N lies in the same interval (in fact it is optimal to have $v^N = v$ if $k > 1$), and v^E is in the adjacent interval above. For v on one of the kinks, it must be that $v^N = v$ and v^E is on the kink above. Therefore for any value above v_n, continuation value are also above v_n. Why does the optimal policy deliver high value after effort? Since the high type's incentive constraint binds,

$$m = \frac{(1-\delta)(1-u_H)}{1-\delta}$$

setting v^E higher lowers the monitoring cost today. At the same time, it lowers the continuation payoff for the principal after observing effort since F is strictly decreasing in this range. However, lowering the monitoring probability means that the expected time effort is observed goes up, so overall offering higher rewards after effort is good for the principal.

In fact, for $v \geq v_n$, the high type and the low type receive the same value: as the high type's incentive constraint binds whenever he exerts effort, he weakly prefers to shirk always, and the low type always shirks. As the only difference in payoffs between types comes when they exert effort, they get the same payoff. The principal and the high type's payoffs, $(F(v), v)$ are on the Pareto frontier of the relationship between the principal and the high type.

To deliver lower values than v_n to the low type, it is no longer possible to use monitoring efficiently, and the relationship is not Pareto efficient. The only way to deliver such low values is to increase monitoring frequency. For $v < v_{n+1}$, the high type's incentive constraint slackens [22] and there is a region in which both the low type and the high type have slack incentives. The low type strictly prefers to shirk, and the high type strictly prefers to exert effort. This is because in this region the principal sets $v^E = v_n$, which is the optimal point for the principal. Setting $v^N = v$ is optimal, and the implied monitoring probability to deliver v to the low type satisfies

$$\frac{(1-\delta)(1-u_H)}{1-\delta+\delta v_n} < m < \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v_n}$$

so there is slack in both incentive constraints. As we lower v, the low type's incentive constraint eventually binds when we hit v_{n+2}, and we have to begin to lower v^E from v_n. For $v \leq v_{n+2}$, $v \in [v_k, v_{k-1}]$, the low type's incentive constraint binds, and the optimal policy

[22] For $v \in [v_{n+1}, v_n]$, the high type's incentive constraint continues to bind.

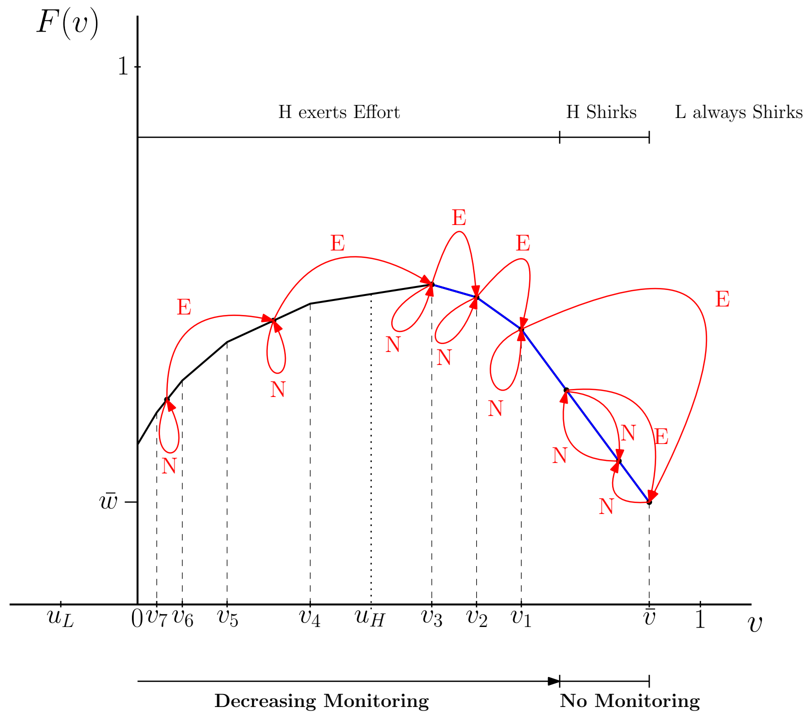


Figure 4: Optimal policy and dynamics.

satisfies v^N in the same interval and v^E two intervals above (again there is an optimal policy that sets $v^N = v$). Therefore for values below v_n, the policy begins inefficiently but the continuation contract will eventually hit the Pareto frontier. Again, delivering higher values after effort is good for incentives, and in this region it is good for the principal's continuation payoffs too as the function is strictly increasing.

The monitoring probability decreases as the principal delivers higher and higher values. The easiest way to see this is that for any $v < v_1$, it is optimal to set $v^N = v$. Since the low type is shirking, promise-keeping for the low type implies

$$m = \frac{(1-\delta)(1-v)}{1-\delta+\delta v}$$

which is decreasing in v. Idea of the proof:
The proof is constructive. I guess what the optimal policy should look like and construct the function above, and then verify that it is optimal by checking that it is the fixed point of the contraction map defined by the problem. Broadly, we can solve for the function in two separate pieces: a region where the high type's incentives bind, and a region where the low type's incentives bind. First we solve for the Pareto frontier of the relationship between

the principal and the high type. I show that if you begin the relationship on the Pareto frontier, it always stays on the Pareto frontier. Therefore for values the high type receives on the Pareto frontier, we can deliver the same value to the low type by having him shirk everywhere. Since the high type always receives weakly higher values than the low type and the principal's payoff is decreasing in high type value, the Pareto optimal policy that delivers the high type a value v is also the optimal policy to deliver that value to the low type.

To construct the function below the Pareto frontier, we know that the high type's incentive constraint will have to slacken eventually. We can guess that there will be some point at which the low type's incentive constraint begins to bind. This allows us to construct the function recursively below the Pareto frontier, guessing what happens in the region where the incentives are slack - the principal sets continuation values after effort to his optimal point on the Pareto frontier.

5.3 Contract Optimisation

Returning to the optimal choice of policy at the screening stage, let σ be an optimal contract that is simple.

Proposition 10. *There exists $\delta^* \in (0,1)$ such that if $\delta > \delta^*$, there is an optimal contract that is simple with Phase 1 monitoring probability m^* and which delivers Phase 2 value v to the low type, such that for every h in Phase 1, the low type's incentive constraint to shirk binds:*

$$m^* = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v}$$

and the principal's payoff from the contract is

$$W(\sigma) = p_0 F(v_L) + (1-p_0)(1-v_L)(\delta\bar{w} - c(1-\delta)) =: W(v_L)$$

where $v_L := V_L(\sigma)$

Proof. See Appendix. □

$W(v_L)$ has generically has a unique maximum on one of the kinks of the function F. As long as the players are sufficiently patient, the maximum occurs on one of the kinks strictly lower than v_{n+1}, and Phase 2 of the contract begins with a value strictly below the Pareto frontier.

Proposition 11. *There exists $\delta^* \in (0,1)$ such that if $\delta > \delta^*$, there is an optimal contract σ that is simple which gives low type value $V_L(\sigma) = v_{n^*}$ for some $n^* \in \{n+2, ..., K\}$, where v_{n^*} is the generically unique maximiser of $W(v_L)$. The contract has Phase 1 monitoring probability*

$$m^* = \frac{(1-\delta)(1-v_{n^*})}{1-\delta+\delta v_{n^*}}$$

and optimally deliver a Phase 2 value to the low type of v_{n^*-2}.

Proof. See Appendix. □

Theorem 3 is an immediate consequence of reading off the optimal policy to deliver v_{n^*-2} in Phase 2.

5.4 No Commitment

Suppose the principal has no commitment. I focus on principal-best PPBE of the game. The best equilibrium for the principal is as follows: the principal continues the relationship and monitors with probability one as long as in every period in the past, the principal has monitored and there has been no shirking. Otherwise, the principal fires the agent (and does not monitor). The high type exerts effort as long as in every period in the past, the principal has monitored and there was no shirking. Otherwise he shirks. The low type shirks at every history. Conditional on the agent being the high type, on-path behaviour has monitoring and effort in every period forever, supported by deviation to a grim trigger strategy of shirking and firing. Conditional on the agent being the low type, he is screened in the very first period, but monitoring costs are very high.

Proposition 12. *There exists $\delta^* \in (0,1)$ such that if $\delta > \delta^*$, there is an equilibrium σ such that for all h,*

$$i) \quad d(h) = \begin{cases} 1 & \textit{if neither } S \textit{ nor } N \textit{ has occurred in the past} \\ 0 & \textit{if } S \textit{ or } N \textit{ has occurred in the past} \end{cases}$$

$$ii) \quad m(h) = \begin{cases} 1 & \textit{if neither } S \textit{ nor } N \textit{ has occurred in the past} \\ 0 & \textit{if } S \textit{ or } N \textit{ has occurred in the past} \end{cases}$$

$$iii)\ s_H(h) = \begin{cases} 0 & \textit{if neither } S \textit{ nor } N \textit{ has occurred in the past} \\ 1 & \textit{if } S \textit{ or } N \textit{ has occurred in the past} \end{cases}$$

$iv)\ s_L(h) = 1.$

σ is a principal-optimal equilibrium and

$$W(\sigma) = p_0(1 - c) + (1 - p_0)(\delta\bar{w} - c(1 - \delta))$$

Proof. See Appendix. □

Without commitment, monitoring is a relatively blunt tool. In any period, the principal needs to monitor with positive probability to incentivise effort from the high type. As monitoring must be a best response, the principal is pays the cost of monitoring in expectation any period that the high type exerts effort. Thus he can do no better with the high type than by monitoring in every period to incentivise effort. With this much monitoring, the low type finds it optimal to shirk and be caught immediately.

6 Remarks

The Role of Transfers

The baseline model does not allow for monetary transfers between players. How important is this assumption to the results of the model, and in what circumstances is this a reasonable assumption? Consider the case of self-enforcing transfers (without commitment) from the principal to agent (limited liability). If we allow for this in the model, then for the pure adverse selection case, nothing changes and transfers are never used. This straightforward (hence I omit the proof) as the principal cannot credibly pay the low type to reveal himself: once the principal learns the agent is the low type, he will fire him and renege on any agreement to pay him. At the same time, there is no need to pay the high type anything since incentives are perfectly aligned. For the case of moral hazard and adverse selection, transfers still play no role in screening for the same reason as above. It is easy to show that the two phase structure of the contract remains, with no transfers from the principal to the low type. Where transfers may play a role is in Phase 2 of contract, where the principal can reward the high type for effort. While I have not explicitly solved for the optimal Phase 2 contract in this case, I conjecture that the structure will be similar to that without transfers. If rewarding the high type with a bonus instead of by letting him shirk is more cost-effective, then the contract will look the same initially, and eventually begin to pay the agent a bonus each time he is observed exerting effort.

If the principal could commit to transfers to the agent then he could pay the low type to leave. However, such transfers may well be infeasible in many circumstances. If we think it likely that the principal cannot commit to not fire the agent (which we can think of as an inflexible transfer), it seems natural to consider cases in which he cannot commit to payments either.

In which applications are transfers more likely to be available to the principal? Investors may indeed have the flexibility to offer rewards for high effort from the companies they fund. If we imagine that good companies and investors have well-aligned preferences, the results suggest that such rewards are not useful in these relationships. In organisations that face moral hazard even from good employees, monetary rewards may be infeasible. In many organisations, the people in charge of monitoring and firing agents are not in a position to embellish their employees' salaries with bonuses. This kind of wage rigidity can often be imposed due to issues of morale and the need to maintain the same standards for all workers. For instance, an agent who is monitored exerting effort is rewarded, while another was not observed and hence not rewarded. This can lead to workers feeling unfairly treated and a deterioration of the work environment.

Full Commitment

If the principal can fully commit both the firing rule and the monitoring policy, then he can offer screening contracts to the agent. The main difference that arises here is that by separating in this way, the principal immediately learns the agent's type, and monitoring is no longer needed for learning. The optimal contract will offer a contract to the high type which looks very similar to Phase 2 of the optimal contract with limited commitment. The low type's contract involves no monitoring. The low type will be employed and allowed to shirk for some time and then will be fired at some point. The duration of employment is just enough to make the low type willing to take this contract rather than the high type's contract. Commitment power lets the principal not fire the low type immediately and learn at time zero, which saves on the costs of monitoring the low type. I omit the proofs as they are analogous to those for the limited commitment case.

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7 Appendix A: Pure Adverse Selection

7.1 Optimal Contract

Proof of existence of optimal contract in Theorem 1:

Proof. I will show that the set of payoffs achievable with a contract, $\mathcal{E}$ is a compact set and therefore that the maximum principal payoff in this set is well-defined.

$$\mathcal{E} := \{\big(W(\sigma), V_L(\sigma)\big) |\ \sigma \text{ is a contract } \}$$

This is a subset of $\mathbb{R}^2$ and is bounded because $\delta < 1$ and the game payoffs are bounded. It is also non-empty, since the profile which specifies $m(h) = 0$ and $s_L(h) = 1$ for all h is a contract with payoffs $(p_0, 1)$. We need to show is that $\mathcal{E}$ is closed.
Let $\{x^n\}$ be a sequence in $\mathcal{E}$ such that $x^n \to x$. By definition of $\mathcal{E}$, for every n, x^n is generated by a contract $\sigma^n = (m^n, s_L^n)$. Now for all n, and for all $h \in \mathcal{H}$, $\sigma^n(h) := (m^n(h), s_L^n(h)) \in [0,1]^2$. By compactness of $[0,1]^2$, the sequence has a convergent subsequence, so passing onto the subsequence, $\sigma^n(h) \to \sigma(h) = (m(h), s_L(h))$ for some $\sigma(h) \in [0,1]^2$.
We want to show that the profile σ defined by this limit is a contract with payoff x. Note that $W(\sigma^n)|_h \to W(\sigma)_h$ and $V_L(\sigma^n)|_h \to W(\sigma)|_h$ for all h: since $\delta < 1$, there exists some T sufficiently large such that the expected payoff after T is less than ϵ. Since $\sigma^n(h^t) \to \sigma(h^t)$ for all h^t, $t < T$, choose N sufficiently large such that for each $t < T$, for $n > N$, $\sigma^n(h^t)$ is sufficiently close to $\sigma(h^t)$.
It follows that

$$\big(W(\sigma), V_L(\sigma)\big) = x$$

To check that σ is a contract, we need to check that agent incentives are satisfied at every history. Fix an arbitrary history h. There are 3 cases:
1) $s_L(h) = 1$. Suppose, for the sake of contradiction that the agent's incentives fail, so

$$m(h)\big(1 - \delta + \delta V_L(\sigma)|_{hE}\big) > (1 - \delta)(1 - u_L)$$

As $V_L(\sigma^n)_{hE} \to V_L(\sigma)|_{hE}$, and since $\sigma^n \to \sigma$, this implies that there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$m^n(h)\big(1 - \delta + \delta V_L(\sigma^n)|_{hE}\big) > (1 - \delta)(1 - u_L)$$

and, $s^n(h) \geq 1 - \epsilon$. For $s^n(h) \geq 1 - \epsilon$ to be incentive compatible for the agent in σ^n, it needs to be that

$$m^n(h)\big(1-\delta+\delta V_L(\sigma^n)|_{hE}\big) \leq (1-\delta)(1-u_L)$$

contradicting that σ^n is a contract.
Case 2: $s_L(h)=0$. The proof is analogous to case 1.
Case 3: $s_L(h) \in (0,1)$. Going along the same lines in the first two cases, suppose for the sake of contradiction that incentives fail, so in particular the agent's indifference condition for mixing does not hold, so he strictly prefers to exert effort (shirk):

$$m(h)\big(1-\delta+\delta V_L(\sigma)|_{hE}\big) > (<)(1-\delta)(1-u_L)$$

By the same argument as above, it must be then that for n sufficiently large,

$$m^n(h)\big(1-\delta+\delta V_L(\sigma^n)|_{hE}\big) > (<)(1-\delta)(1-u_L)$$

while $s_L^n(h) \in (s_L(h)-\epsilon, s_L(h)+\epsilon) \subset (0,1)$. This contradicts that σ^n is a contract. Therefore σ is a contract and $\mathcal{E}$ is a closed and bounded subset of $\mathbb{R}^2$. Therefore,

$$w^* = \max\{w|(w,v_L) \in \mathcal{E}\}$$

is well-defined and an optimal contract exists. □

Proof of Proposition 2: We begin by constructing the sequence $\{v_k\}$.

Definition 5. *Define the sequence $\{v_k\}_{k=0}^{\infty}$ as the solution to the recursion $v_0 := 1$, and*

$$v_k = (1-m)\left(1-\delta+\delta v_k\right)$$

and

$$m = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v_{k-1}}$$

Observation 1. *The solution to the above recursion is given by $\{v_k\}_{k=0}^{\infty}$ where*

$$v_k = \frac{\delta^k(1-u_L)^k + u_L\sum_{i=0}^{k-1}\delta^i(1-u_L)^i}{\sum_{i=0}^{k}\delta^i(1-u_L)^i}$$

and
i) v_k is strictly decreasing.
ii) $\lim_{k\to\infty} v_k = \max\{u_L, -\frac{1-\delta}{\delta}\}$.

Define the truncated sequence $\{v_k\}_{k=0}^{K}$ where $K-1$ is defined as the highest k such that $v_k > 0$, let the new sequence coincide with the original sequence from 0 to $K-1$, and let $v_K = 0$.

Lemma 1. *Let $v \in [v_k, v_{k-1})$ and let PK hold and IC bind. Then $v^N = v$ if and only if $v^E \in [v_{k-1}, v_{k-2}]$.*

Proof. By the recursive definition of the sequence, if $v = v_k$, PK holds and IC binds, then $v^E = v_{k-1}$. IC binds and PK holds, then substituting for m into IC using PK,

$$v^E = \frac{(1-\delta)(1-u_L-m)}{\delta m} = \frac{v(1-u_L)}{1-v} - \frac{1-\delta}{\delta} =: v^E(v)$$

Therefore if IC binds and PK holds, v^E is increasing in v, satisfies $v^E(v_k) = v_{k-1}$, $v^E(v_{k-1}) = v_{k-2}$, so for $v \in (v_k, v_{k-1})$, $v^E(v) \in [v_{k-1}, v_{k-2}]$. If $v \notin [v_k, v_{k-1}]$ then clearly $v^E(v) \notin [v_{k-1}, v_{k-2}]$. □

Defined the function $\hat{F}$ as

$$\hat{F}(v) := 1 - c(1-\delta)\sum_{i=0}^{k-1}\delta^i(1-u_L)^i\left(1+(k-1-i)u_L\right) + c(1-\delta)\sum_{i=0}^{k-1}(k-i)v$$

for $v \in [v_k, v_{k-1}], k \in \{0, .., K\}$

Observation 2. *$\hat{F}$ is strictly increasing, piecewise linear and concave.*

Proof. The fact that it is strictly increasing and piecewise linear readily follows from the definition, and concavity follows by noting that

$$\hat{F}(v) = \min_{k\in\{0,..,K\}} \hat{F}_k(v)$$

where

$$\hat{F}_k(v) := 1 - c(1-\delta)\sum_{i=0}^{k-1}\delta^i(1-u_L)^i\left(1+(k-1-i)u_L\right) + c(1-\delta)\sum_{i=0}^{k-1}(k-i)v$$

□

We wish to show that $\hat{F}$ is the fixed point of the operator T. To that end, we begin by proving some properties of the operator when applying it to the candidate. The next lemma says that if the agent is exerting effort, then his incentive constraint must bind.

Lemma 2. *Let γ be an optimal policy at $T\hat{F}(v)$. If $s_L = 0$ then IC must bind. Therefore $s_L = 1$ without loss of generality.*

Proof. Suppose for the sake of contradiction that $s_L = 0$ and the IC is slack. I will find an improvement, contradicting the optimality of γ. The IC slack means that

$$m > \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E}$$

Define the new policy γ': $m' = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E}$, $s'_L = 0$, $v^{N'} = \frac{v-(1-\delta)u_L-\delta m' v^E}{\delta(1-m')}$, $v^{E'} = v^E$. Then clearly $m' < m$, and IC is satisfied. The agent's payoff from the new policy is

$$(1-\delta)u_L + \delta(1-m')v^{N'} + \delta m' v^E = v$$

so PK holds. Define $\lambda := \frac{1-m}{1-m'}$ and notice that

$$\begin{aligned}
\lambda v^N + (1-\lambda)v^E &= \frac{1-m}{1-m'}\frac{v-(1-\delta)u_L - \delta m v^E}{\delta(1-m)} + \frac{m-m'}{1-m'}v^E \\
&= \frac{1}{\delta(1-m')}\left(v-(1-\delta)u_L - \delta m v^E + \delta m v^E - \delta m' v^E\right) \\
&= v^{N'}
\end{aligned}$$

The principal's payoff gain from the new policy is

$$(1-\delta)(m-m')c + \delta\left[(1-m')\hat{F}(v^{N'}) + m'\hat{F}(v^E) - (1-m)\hat{F}(v^N) - m\hat{F}(v^E)\right]$$

which equals

$$(1-\delta)(m-m')c + \delta(1-m')\left[\hat{F}(v^{N'}) - \frac{1-m}{1-m'}\hat{F}(v^N) - \frac{m-m'}{1-m'}\hat{F}(v^E)\right]$$

which is the same as

$$(1-\delta)(m-m')c + \delta(1-m')\left[\hat{F}(v^{N'}) - \lambda\hat{F}(v^N) - (1-\lambda)\hat{F}(v^E)\right] > 0$$

where the inequality follows from the concavity of $\hat{F}$. Therefore γ' is an improvement, contradicting the optimality of γ. As the agent's actions do not enter the principal's payoff since the low type is not there on path, we can set the agent's action to $s_L = 1$ without loss, since the agent is indifferent and receives the same payoff from shirking or effort. □

Given this result, we know that if we are applying T to $\hat{F}$, the agent can always be shirking without loss of generality since if he is exerting effort, he is indifferent. This reduces T to:

$$
\begin{aligned}
T\hat{F}(v) = \max_{m\in[0,1],v^N,v^E\in[0,1]} \quad & (1-\delta)(1-mc) + \delta(1-m)\hat{F}(v^N) + \delta m\hat{F}(v^E) \\
\text{subject to} \quad & v = (1-m)[1-\delta+\delta v^N] && (PK) \\
& m \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E} && (IC)
\end{aligned}
$$

I now show that $\hat{F}$ is indeed the fixed point of the reduced operator and therefore solves the principal's problem.

Claim 1. *$T\hat{F} = \hat{F}$.*

Proof. For any policy γ, define

$$J(\gamma) := (1-\delta)(1-mc) + \delta(1-m)\hat{F}(v^N) + \delta m\hat{F}(v^E)$$

and for $k \in \mathbb{N}$,

$$J_k(\gamma) := (1-\delta)(1-mc) + \delta(1-m)\hat{F}_k(v^N) + \delta m\hat{F}_{k-1}(v^E)$$

where we define $\hat{F}_0(v) := 1$. Since $\hat{F} \leq F_k$ for all k,

$$J_k(\gamma) \geq J(\gamma)$$

for all k, with equality if $v^N \in [v_k, v_{k-1}], v^E \in [v_{k-1}, v_{k-2}]$. Consider the collection of relaxed programs

$$
\begin{aligned}
F_k^*(v) = \max_{\gamma} \quad & J_k(\gamma) \\
\text{subject to} \quad & v = (1-m)[1-\delta+\delta v^N] && (PK)
\end{aligned}
$$

$$m \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E} \qquad (IC)$$

Then clearly $F_k^*(v) \geq T\hat{F}(v)$ for all $v \in [\underline{v}, 1]$, for all k. Let $v \geq v_1$. For any γ such that PK and IC hold,

$$\begin{aligned} J_1(\gamma) &= (1-\delta)(1-mc) + \delta(1-m)\hat{F}_1(v^N) + \delta m \hat{F}_0(v^E) \\ &= (1-\delta)(1-mc) + \delta(1-m)\left(1 - c(1-\delta)(1-v^N)\right) + \delta m \\ &= (1-\delta)(1-mc) + \delta(1-m) - c(1-\delta)(1-m-v) + \delta m \\ &= 1 - c(1-\delta)(1-v) = \hat{F}_1(v) \end{aligned}$$

where we use the fact that $\hat{F}_0(v^E) = 1$ and from PK, $v^N = \frac{1-m-v}{\delta(1-m)}$. so $F_1^*(v) = \hat{F}_1(V)$. If we can find a policy γ such that $v^E = 1$, $v^N \in [v_1, 1]$ and PK and IC hold, then $F_1^*(v)$ is achievable in $T\hat{F}(v)$ so this must be optimal.
Take $v^N = v$, $v^E = 1$, and $m = \frac{(1-\delta)(1-v)}{1-\delta+\delta v}$, which comes from PK, so clearly PK holds. To check that IC holds, we need

$$m = \frac{(1-\delta)(1-v)}{1-\delta+\delta v} \leq (1-\delta)(1-u_L)$$

Since $v \geq v_1$, we have that

$$\begin{aligned} \frac{(1-\delta)(1-v)}{1-\delta+\delta v} &\leq \frac{(1-\delta)(1-v_1)}{1-\delta+\delta v_1} \\ &= (1-\delta)(1-u_L) \end{aligned}$$

so IC holds. Therefore, $T\hat{F}(v) = F_1(v)$.
Let $v \in [0, v_1)$, such that $v \in [v_k, v_{k-1})$ for some $k \geq 1$. To simplify notation, rewrite F_k as $F_k(v) = A_k + vB_k$, where

$$A_k = 1 - c(1-\delta)\sum_{i=0}^{k-1} \delta^i (1-u_L)^i (1 + (k-1-i)u_L)$$

and

$$B_k = c(1-\delta)\sum_{i=0}^{k-1} \delta^i (1-u_L)^i (k-i)$$

Note that

$$A_k - A_{k-1} = -c(1-\delta)\delta^{k-1}(1-u_L)^{k-1} - c(1-\delta)u_L \sum_{i=0}^{k-2} \delta^i (1-u_L)^i$$

and

$$B_k - B_{k-1} = c(1-\delta)\sum_{i=0}^{k-1} \delta^i (1-u_L)^i$$

For any γ such that PK and IC hold,

$$\begin{aligned}
J_k(\gamma) =& (1-\delta)(1-mc) + \delta(1-m)\hat{F}_k(v^N) + \delta m \hat{F}_{k-1}(v^E) \\
=& (1-\delta)(1-mc) + \delta(1-m)\left(A_k + v^N B_k\right) + \delta m\left(A_{k-1} + v^E B_{k-1}\right) \\
\leq& (1-\delta)(1-mc) + \delta A_k - \delta m(A_k - A_{k-1}) + \big(v - (1-m)(1-\delta)\big)B_k + (1-\delta)(1-u_L-m)B_{k-1} \\
=& (1-\delta)(1-mc) + \delta A_k - \delta m(A_k - A_{k-1}) - (1-m)(1-\delta)(B_k - B_{k-1}) - (1-\delta)u_L B_{k-1} + vB_k \\
=& (1-\delta)(1-mc) + \delta A_k + \delta m\left(c(1-\delta)\delta^{k-1}(1-u_L)^{k-1} + c(1-\delta)u_L\sum_{i=0}^{k-2}\delta^i(1-u_L)^i\right) \\
& - (1-m)(1-\delta)^2 c \sum_{i=0}^{k-1}\delta^i(1-u_L)^i - (1-\delta)^2 c u_L \sum_{i=0}^{k-2}\delta^i(1-u_L)^i(k-1-i) + vB_k \\
=& \delta A_k + (1-\delta)\underbrace{\left(1 - c(1-\delta)\left(\sum_{i=0}^{k-1}\delta^i(1-u_L)^i - u_L\sum_{i=0}^{k-2}\delta^i(1-u_L)^i(k-1-i)\right)\right)}_{=A_k} \\
& + mc(1-\delta)\underbrace{\left(-1 + \delta^k(1-u_L)^{k-1} + u_L\sum_{i=0}^{k-2}\delta^{i+1}(1-u_L)^i + (1-\delta)\sum_{i=0}^{k-1}\delta^i(1-u_L)^i\right)}_{=0} + vB_k \\
=& A_k + vB_k = \hat{F}_k(v)
\end{aligned}$$

with equality only if IC binds. Therefore $F^*_k(v) \leq \hat{F}_k(v)$.

If there exists a policy γ such that PK holds and IC binds, with $v^N \in [v_k, v_{k-1}], v^E \in [v_{k-1}, v_{k-2}]$, then $\hat{F}_k(v)$ is achievable at $T\hat{F}(v)$.

Take γ such that $v^N = v$, $m = \frac{(1-\delta)(1-v)}{1-\delta+\delta v}$ from PK, and v^E such that $m = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E}$ so that IC binds. By Lemma 1, $v^E \in [v_{k-1}, v_{k-2}]$. Therefore,

$$T\hat{F}(v) = \hat{F}_k(v) = F(v)$$

□

To prove ii), iii) and iv):

Claim 2. *Let γ be an optimal policy at v. Then*
i) $v \in [v_k, v_{k-1}]$ implies $v^N \in [v_k, v_{k-1}], v^E \in [v_{k-1}, v_{k-2}]$.
ii) $v \leq v_1$ implies that IC binds.
iii) $v = v_k$ for $k \geq 1$ implies $v^N = v_k$, $v^E = v_{k-1}$, and $m = \frac{(1-\delta)(1-v_k)}{1-\delta+\delta v_k}$

Proof. 1) and ii) follow directly from the proof of claim 1. For ii), note that v_k satisfies that setting $v^E = v_{k-1}$ and having IC bind results in $v^N = v_k$ from PK:

$$v_k = (1-m)[1-\delta+\delta v_k]$$

with $m = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v_{k-1}}$.
Let $v = v_k$. Suppose $v^E > v_{k-1}$. Then IC implies that the monitoring probability m' satisfies

$$m' < \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v_{k-1}}$$

To satisfy PK, we need v^N such that

$$v_k = (1-m')\left(1-\delta+\delta v^N\right)$$

which is only feasible if $v^N < v_k$ since $m' < m$. This violates the optimality condition in i). Therefore it must be that $v^E = v_{k-1}$, and the fact that IC binds implies that $v^N = v_k$ and $m' = m$. □

This completes the proof of Proposition 2.

Proof of proposition 3:

Proof. Let $W(\sigma) := w$. To show that the agent's incentive to shirk binds, suppose for the sake of contradiction that it is slack, so

$$m^* < \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v}$$

The principal's payoff is

$$w = m^*\Big(\delta\big(p_0 F(v) + (1-p_0)\big)\bar{w} - c(1-\delta)\Big) + (1-m^*)\big((1-\delta)p_0 + \delta w\big)$$

By the proof of proposition 1, if $\delta > \frac{c}{c+\bar{w}(1-p_0)}$ then $w > p_0$, so it must be that the term multiplied by m^* is strictly higher than w. If $m^* < 1$, the principal can increase it slightly, maintain incentives and improve his payoff, a contradiction. If $m^* = 1$, then since $F(v)$ is strictly increasing in v, the principal raise v slightly without affecting incentives and improve his payoff, a contradiction. Therefore it must be that the agent is indifferent.
For the second part, notice from the equation above that

$$w = p_0 \frac{(1-\delta)(1-m^*c) + \delta m^* F(v)}{1-\delta(1-m^*)} + \frac{m^*}{1-\delta(1-m^*)}(1-p_0)\big(\delta\bar{w} - c(1-\delta)\big)$$

Since the agent is shirking and σ is a simple contract,

$$v_L = (1-m^*)(1-\delta+\delta v_L)$$

and since the agent's incentive constraint binds,

$$m^* = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v}$$

Now $v \in [v_k, v_{k-1})$ for some k. By Lemma 1, the above two equalities imply that $v_L \in [v_{k+1}, v_k]$. Therefore, as in that proof, substituting in for v from the incentive constraint,

$$\frac{(1-\delta)(1-m^*c) + \delta m^* F(v)}{1-\delta(1-m^*)} = F(v_L)$$

Substituting in for m^* in terms of v_L,

$$\frac{m^*}{1-\delta(1-m^*)} = (1-v_L)$$

and the result follows. □

Proof of Proposition 4:

Proof. By proposition 3, an optimal contract that is simple chooses $v_L \in [0,1]$ to maximise $W(v_L)$.

Since F is piecewise liner, W continuous and differentiable almost everywhere and the right and left derivative of W exists everywhere. By continuity of W, by Weierstrass' Theorem, there exists $v^* \in [0,1]$ tht maximises W. Furthermore, v^* must satisfy

$$W'_-(v^*) \geq 0 \geq W'_+(v^*)$$

where $W'_{-/+}$ is the left/right derivative of W.
Let v^* be a solution. Then the first order condition is that

$$p_0 F'_-(v^*) - (1-p_0)\big(\delta\bar{w} - c(1-\delta)\big) \geq 0 \geq p_0 F'_+(v^*) - (1-p_0)\big(\delta\bar{w} - c(1-\delta)\big)$$

or

$$F'_-(v^*) \geq \frac{1-p_0}{p_0}\big(\delta\bar{w} - c(1-\delta)\big) \geq F'_+(v^*)$$

Since F is concave and piecewise linear, if both inequalities above are strict, then v^* is the unique solution and $v^* = v_n$ for some n since for $v_L \in (v_k, v_{k-1})$, F is differentiable with constant derivative. Generically, this will be the case, since if there is an equality in either of the above, a slight perturbation of the parameter c upwards or downwards will make both inequalities strict. Furthermore, if $\delta > \frac{c}{c+\bar{w}(1-p_0)}$, $v^* \geq 1$ since for $v_L \geq v_1$,

$$F'_-(v_L) = c(1-\delta) < \frac{1-p_0}{p_0}\big(\delta\bar{w} - c(1-\delta)\big)$$

Therefore the necessary first order condition cannot be satisfied for $v_L > v_1$.
Let $V_L(\sigma) = v_n$ be the optimal choice of agent value. By definition of a simple contract, the agent must shirk in Phase 1, so for all h in Phase 1, for which $n(h) = 0$,

$$m(h) = \frac{(1-\delta)(1-v_n)}{1-\delta+\delta v_n}$$

By proposition 3 the agent's incentive constraint binds, so by Lemma 1, for every h in Phase 1, $V_L(\sigma)|_{hE} = v_{n-1}$. This value must be delivered optimally in Phase 2, and by Proposition 4, the continuation payoffs of the agent must satisfy $V_L(\sigma)|_h = v_{n-n(h)}$ for all h with $n(h) > 0$. Furthermore, the agent shirks in at every history in Phase 2, so

$$m(h) = \frac{(1-\delta)(1-v_{n-n(h)})}{1-\delta+\delta v_{n-n(h)}}$$

To show generic uniqueness: suppose there exists a unique optimal contract that is simple. Let σ be an optimal contract. Suppose for the sake of contradiction, that σ does not have the form of the unique simple contract. That is, there exist distinct Phase 1 histories h and h' such that $m(h) \neq m(h')$ (since the incentive constraint must bind in Phase 1, this is necessary and sufficient for the contract to not be simple).
By the proof of Proposition 1, the contracts $\sigma|_h$ and $\sigma|_{h'}$ are both optimal contracts and the simple contract constructed by specifying σ_h at every history in Phase 1, and the simple contract constructed by specifying $\sigma_{h'}$ are distinct optimal contracts, hence distinct optimal simple contracts, a contradiction. Therefore there exists a unique optimal contract. □

Proof of Theorem 1: The result is a corollary of proposition 1, 2 and 4. To construct the sequence, note that for any history such that $n_h = 0$, we are in Phase 1, and the monitoring probability is as given in proposition 4, $m^* = \frac{(1-\delta)(1-v_{n^*})}{1-\delta+\delta v_{n^*}}$. By proposition 4, Phase 2 begins with the value v_{n^*-1}, and therefore the at any history in Phase 2, the monitoring probability is

$$m(h) = \frac{(1-\delta)(1-v_{n^*-n_h})}{1-\delta+\delta v_{n^*-n_h}}$$

Therefore the sequence of probabilities is strictly decreasing, depends only on n_h, and after the principal has monitored n^* times, monitoring ends.

7.2 No Commitment: proof of theorem 2 and propositions 5-7

For technical convenience, I frame the problem recursively. Define the equilibrium value set at belief p as the set

$$\mathcal{E}(p) := \{(W(\sigma), V(\sigma))|(\sigma) \text{ a PPBE with } p_0 = p\}$$

of equilibrium value pairs. Let $w(p)$, and $v(p)$, the principal and agent's equilibrium value correspondence at p, respectively, be the projections of $\mathcal{E}(p)$ onto the first and second coordinate respectively.

Definition 6 (Equilibrium)**.** *Let $(w, v) \in \mathcal{E}(p)$. Then there exist actions $m \in [0, 1]$, $s_L \in [0, 1]$, and continuation values for both players after each public outcome, w^E, w^N, v^E, v^N, such that $(w^E, v^E) \in \mathcal{E}(p')$, with $p' = \frac{p}{1-s_L(1-p)}$ and $(w^N, v^N) \in \mathcal{E}(p)$, subject to promise-*

keeping and incentive compatibility for both players:

$$w = m\big[\big(1 - s_L(1-p)\big)\big((1-\delta)p' + \delta w^E\big) + s_L(1-p)\delta\overline{w}\big] + (1-m)[(1-\delta)p + \delta w^N] \quad \text{(PPK)}$$

$$v = s_L(1-m)[(1-\delta) + \delta v^N] + (1-s_L)\big[(1-\delta)u_L + \delta(mv^E + (1-m)v^N)\big] \quad \text{(APK)}$$

$$m \in \arg\max_{m' \in [0,1]} \; m'\big[\big(1 - s_L(1-p)\big)\big((1-\delta)p' + \delta w^E\big) + s_L(1-p)\delta\overline{w}\big] + (1-m')[(1-\delta)p + \delta w^N] \quad \text{(PIC)}$$

$$s_L \in \arg\max_{s' \in [0,1]} \; s'(1-m)[(1-\delta) + \delta v^N] + (1-s')\big[(1-\delta)u_L + \delta(mv^E + (1-m)v^N)\big] \quad \text{(AIC)}$$

I will characterise the equilibrium value set at every belief. Recall that given a belief p, this is a lower bound on the principal's equilibrium payoff. That is, if $w \in w(p)$, then $w \geq p$. It must also be that for $v \in v(p)$, $v \geq 0$ as the agent can always shirk and at worst be fired and get his outside option, so 0 is a lower bound on the agent's equilibrium payoff.

I first describe the general steps taken for the proof before going into the details. Outline of proof:

1. If the belief is $p > \overline{p}$, the principal never monitors and the agent always shirks. If the belief is at $\overline{p}$, it must be that the agent shirks with probability 1. This is a best response for the agent only if the the principal monitors with sufficiently low probability. If the belief is $p < \overline{p}$ the agent must strictly mix between his actions. The agent's indifference condition pins down the principal's monitoring probability.

2. If the principal receives value $w > p$ at belief p, it must be that the agent receives value $v = 0$: in order to benefit from monitoring, the principal must strictly prefer to monitor while the agent shirks with positive probability, which gives the the agent a payoff of 0. In order to deliver the agent 0 at p, it must be that p' is such that the specific value $v_0^E \in v(p')$.

3. It is impossible to deliver the agent a continuation value v^E or v^N of 0. Therefore if the principal receives value w at belief p, then his continuation values satisfy $w^N = p$ and $w^E = p'$. That is, there are no gains from monitoring after time 0. This proves proposition 2.

4. Suppose the principal's payoff is w at $p < \overline{p}$. If the principal monitors, the belief moves by a minimum amount. Therefore the belief crosses $\overline{p}$ after finitely many periods of

monitoring. In particular, $p' \geq \frac{p}{\overline{p}}$, and the inequality is strict if and only if $w > p$. By 3), after time 0 this must hold with equality.

5. Using these properties of equilibrium, we can compute the equilibrium value correspondence of the agent at any belief: There exists a cutoff belief $\overline{p}^{n^*}$ such that $v_0^E \in v(\overline{p}^{n^*})$ and v_0^E is not in the correspondence at any other belief. This is used to show that for any $p < \overline{p}^{n^*+1}$, $v(p) = 0$. The principal monitors with probability 1, and if he does not catch the agent, the belief jumps to $\overline{p}^{n^*}$. The principal's payoff is uniquely determined and there are gains from monitoring. For $p > \overline{p}^{n^*+1}$, the agent's value is always strictly positive, so the principal's payoff must be p.

6. By points 4 and 5, the agent's strategy either satisfies $p' = \frac{p}{\overline{p}}$, or if $p < \overline{p}^{n^*+1}$ then $p' = \overline{p}^{n^*}$. Thus the agent's strategy is Markovian in the belief and the path of beliefs is uniquely determined as stated in proposition 1.

Notice that at any belief, if the principal does not monitor, it is a strict best response for the agent to shirk since this action gives the agent his best payoff and has no effect on continuation payoffs. The principal never monitors once the belief crosses $\overline{p}$ and $\mathcal{E}(p) = \{(p, 1)\}$ for $p > \overline{p}$. All the action occurs for $p \leq \overline{p}$. If the belief is at $\overline{p}$, the principal is indifferent between monitoring and not monitoring when the agent shirks with probability one, and strictly prefers to not monitor otherwise. Therefore at $\overline{p}$ the agent must shirk with probability one. The principal can monitor with at most some probability which makes the agent indifferent between shirking and effort. For $p < \overline{p}$, monitoring is a strict best response for the principal when the agent shirks. Therefore the principal monitors with positive probability. Furthermore, it must be that the probability of shirking is strictly less than one, since otherwise the principal monitors with probability 1, to which the agent has a beneficial deviation. These observations are summarised in the following claim.

Claim 3. *Let* $(w, v) \in \mathcal{E}(p)$*. Then*
i) If $p = \overline{p}$ *then* $s_L = 1$ *and* $m \leq (1 - \delta)(1 - u_L)$*.*
ii) If $p < \overline{p}$ *then* $s_L < 1$ *and* $m > 0$*.*

Proof. i) For the sake of contradiction, suppose $s_L < 1$. By definition of $\overline{p}$, not monitoring is a strict best response for the principal, but then shirking is a strict best response for the agent, so this cannot be an equilibrium. Therefore $s_L = 1$ and this must be incentive compatible for the agent. Shirk is a best response if

$$(1-m)\big((1-\delta)+\delta v^N\big) \geq (1-\delta)u_L + \delta v^E + \delta(1-m)v^N$$

Now noting $p' = 1$ implies $v^E = 1$, this requires

$$m \leq (1-\delta)(1-u_L)$$

ii) To derive a contradiction, suppose $s_L = 1$. As $p < \overline{p}$, monitor is a strict best response so $m = 1$, which implies that $v = 0$. Since $p' = 1$, $v^E = 1$, so by deviating to effort, the agent obtains a payoff of

$$(1-\delta)u_L + \delta > 0$$

a profitable deviation, contradicting that this is an equilibrium. Now suppose $m = 0$. Then it must be that $s_L = 1$. Clearly this is not an equilibrium as monitor is a strict best response for the principal. □

For any belief p, suppose there exists an equilibrium in which the principal's payoff is strictly better than p. The following claim says that the agent's payoff in that equilibrium must be 0. The logic is that if there are to be gains from monitoring, there must be some history at which the principal strictly prefers to monitor when the agent shirks with positive probability, and this means the agent must be receiving a payoff of 0.

Claim 4. *Let $p > 0$ and $(w, v) \in \mathcal{E}(p)$. If $w > p$, then $v = 0$.*

Proof. Let h be the history and σ the corresponding equilibrium. If $w > p$ then it must be that $p < \overline{p}$. Observe that for each path from h which is played with positive probability by the principal, there exists a history h' on that path such that $m(h') = 1$ and this is a strict best response. If there is a path for which no such history exists, then the expected payoff from that path is the same as never monitoring, since not monitoring is always a best response on that path. But since the payoff at h must be equal to the payoff from any path which is played with positive probability from h, this implies that $w = p$, a contradiction. If $m(h') = 1$ is a strict best response, it must be that $p(h') < \overline{p}$. Furthermore, it must be that on each path there exists h' with both $m(h') = 1$ and $s_L(h') > 0$, since otherwise the principal's payoff from that path would be p or lower, a contradiction. Let h' be the first history on each path from h which is played with positive probability with $m(h') = 1$ and $s_L(h') > 0$. It must be that $V(\sigma)|_{h'} = 0$ since shirk is a best response. Take any path which is played with positive probability from h. Suppose $h' \neq h$. At every history h'' between h and h', we have $p(h'') < \overline{p}$. By claim 1, $s_L(h'') < 1$, so the agent is willing to exert effort at every history before h'. Therefore the agent's payoff from the path is a combination of $(1-\delta)u_L$ and 0, which is strictly negative. If $h = h'$, the agent's payoff at h is 0. Therefore, if there exists a path which is played with positive probability from h with $h' \neq h$, the agent's payoff at h is strictly negative, a contradiction. Therefore it must be that $h' = h$ on every path played with positive probability. That is, $m = 1$, $s_L > 0$ and $v = 0$. □

Claim 5. *Let $p \leq \overline{p}$ and $(w, v) \in \mathcal{E}(p)$. Then*
i) If $w = p$ then $w^N = p$.
ii) $w^E = p'$

Proof. i) For the sake of contradiction, suppose $w = p$ and $w^N > p$. Then it must be that $p < \overline{p}$, so by claim 1, $m > 0$ and monitoring must be a best response. However, the payoff from monitoring is p, while the payoff from not monitoring is $(1-\delta)p + \delta w^N > p$, a contradiction.
ii) Observe first, that we must have $v^E > 0$: for $p = \overline{p}$ this is obvious as $m < 1$, so let $p < \overline{p}$. Suppose $v^E = 0$. By claim 1, effort must be incentive compatible for the agent, which requires that

$$m \geq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E} = (1-u_L) > 1$$

which is impossible. Therefore $v^E > 0$, which implies by claim 2, that $w^E = p'$. □

Proposition 2 is a consequence of this result: if σ is an equilibrium, and h is an on-path history strictly after time zero, then the principal's payoff from this history is $p(h)$. Therefore the principal always weakly prefers not to monitor after time zero, and there are no gains from monitoring after time zero.

Proof of Proposition 2. $h^t = h^{t-1}E$ or $h^t = h^{t-1}N$ for some history h^{t-1}. For the first case, by claim 3ii), $W(\sigma)|_{h^t} = W(\sigma)|_{h^{t-1}E} = p(h^t)$. For the second case, if $W(\sigma)|_{h^{t-1}N} > p(h^{t-1})$, then by claim 2, $V(\sigma)|_{h^{t-1}} = 0$, which requires $m(h^{t-1}) = 1$, which implies that $h^{t-1}N$ is not on path, contradicting what we assumed. Therefore it must be that $W(\sigma)|_{h^{t-1}N} = p(h^{t-1})$, so the result follows by claim 3i). □

The next result shows that each time the belief increases in equilibrium, it must move by a minimum amount (that is, the agent's shirking probability is bounded below). Since the principal's expected payoff after monitoring is p', if p' is too low, the principal is not willing to monitor.

Claim 6. *Let $p \leq \overline{p}$ and $(w, v) \in \mathcal{E}(p)$. Then $p' \geq \frac{p}{\overline{p}}$.*

Proof. Observe first that
$$p' \geq \frac{p}{\overline{p}}$$
iff
$$\frac{p}{1 - s_L(1-p)} \geq \frac{p}{\overline{p}}$$
iff
$$s_L \geq \frac{c(1-\delta)}{\delta\overline{w}(1-p)}$$
iff
$$p + s_L(1-p)\delta\overline{w} - c(1-\delta) \geq p$$

By claim 1i), it is true for $p = \overline{p}$, so let $p < \overline{p}$. Now for the sake of contradiction, suppose $p' < \frac{p}{\overline{p}}$. By claim 3, $w^E = p'$, so

$$\begin{aligned} w &= \big(1 - s_L(1-p)\big)\big((1-\delta)p' + \delta w^E\big) + s_L(1-p)\delta\overline{w} - c(1-\delta) \\ &= p + s_L(1-p)\delta\overline{w} - c(1-\delta) \\ &< p \end{aligned}$$

where the last inequality is equivalent to $p' < \frac{p}{\overline{p}}$. This is a contradiction as $w \geq p$. □

A corollary of this is that in equilibrium, eventually either the agent must be caught or the belief must cross $\overline{p}$:

Corollary 2. *Let (σ) be an equilibrium. Then with probability 1, eventually the belief crosses $\overline{p}$ or the agent is fired.*

Proof. Given any $p_0 < \overline{p}$, $p_0 \in (\overline{p}^{n+1}, \overline{p}^n)$ for some $n \geq 1$. By proposition 3, after at most n times of the principal monitoring and not catching the agent, the belief will cross $\overline{p}$. Conditional on not catching the agent, the principal monitors n times with probability 1, since he monitors with strictly positive probability when the belief is below $\overline{p}$. □

By proposition 3 and claim 1, we have that in equilibrium, when the belief is below $\overline{p}$, the agent must be strictly mixing between shirk and effort, so the agent's indifference condition is necessary. This pins down the principal's monitoring probability:

Observation 3. *Let $p < \overline{p}$ and $(w, v) \in \mathcal{E}(p)$. Then $s_L \in (0, 1)$ and*

$$m = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E}$$

The following lemma shows that given a belief p, an equilibrium payoff strictly higher than p is equivalent to a jump in the belief today strictly higher than the minimum.

Lemma 3. *Let $(w, v) \in \mathcal{E}(p)$. Then $w > p$ iff $p' > \frac{p}{\overline{p}}$.*

Proof. ($\Leftarrow$): Assume $p' > \frac{p}{\overline{p}}$. By proof of proposition 3, this is equivalent to

$$p + s_L(1-p)\delta\overline{w} - c(1-\delta) > p$$

Therefore,

$$\begin{aligned} w &= \big(1 - s_L(1-p)\big)\big((1-\delta)p' + \delta w^E\big) + s_L(1-p)\delta\overline{w} - c(1-\delta) \\ &= \big(1 - s_L(1-p)\big)p' + s_L(1-p)\delta\overline{w} - c(1-\delta) \\ &> p \end{aligned}$$

where the first equality is because $w^E = p'$ (by claim 3) and the last inequality from the above observation.
($\Rightarrow$): Let $w > p$. By claim 3 $w^E = p'$. Therefore,

$$\begin{aligned} p + s_L(1-p)\delta\overline{w} - c(1-\delta) &= \big(1 - s_L(1-p)\big)p' + s_L(1-p)\delta\overline{w} - c(1-\delta) \\ &= \big(1 - s_L(1-p)\big)\big((1-\delta)p' + \delta p'\big) + s_L(1-p)\delta\overline{w} - c(1-\delta) \\ &= w > p \end{aligned}$$

which implies $p' > \frac{p}{\overline{p}}$. □

I now proceed to fully characterise the players' equilibrium value correspondences. In preparation for this I introduce some new definitions. In order for the principal to receive a payoff in equilibrium higher than p at $p < \overline{p}$, it must be that the agent receives a value of 0. That is $0 \in v(p)$. This implies that $m = 1$ since otherwise the agent could guarantee a positive payoff by shirking, so promise keeping says

$$0 = (1-\delta)u_L + \delta v^E$$

which requires

$$v^E = \frac{(1-\delta)(-u_L)}{\delta} =: v_0^E$$

and we define this value as v_0^E, the continuation value needed to deliver the agent a payoff of 0 in equilibrium. Therefore, delivering 0 at p is feasible if there exists some $p' \geq \frac{p}{\overline{p}}$ with $v_0^E \in v(p')$.

Remark 1. *The preceding analysis has significantly reduced what can occur in equilibrium. In summary, for $p < \overline{p}$, if $(w, v) \in \mathcal{E}(p)$, it must be that:*
1) $m = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E}$ and $w = p + (1 - \frac{p}{p'})\delta\overline{w} - c(1-\delta)$.
2) $p' \geq \frac{p}{\overline{p}}$ and if $p' > \overline{p}$ then $v = 0$ and $v^E = v_0^E \in v(p')$.

where for 1) I am using the claim 3 and the fact that $s_L(1-p) = 1 - \frac{p}{p'}$.

I define the sequence $\{v_n\}$ as

$$v_n := \frac{\delta^n(1-u_L)^n + u_L \sum_{i=0}^{n-1} \delta^i (1-u_L)^i}{\sum_{i=0}^{n} \delta^i (1-u_L)^i}$$

with the convention that $v_0 = 1$. Generically this function defines the agent's equilibrium value at any belief at which delivering the agent 0 is not feasible. It turns out that for n such that $v_n > 0$, it must be that the agent's equilibrium value is strictly positive when $p \geq \overline{p}^n$. Define[23]

$$n^* := \{n \in \mathbb{N} | \; v_{n+1} < 0 < v_n\}$$

Then $n^* \geq 2$ since $v_1 > 0$ and $\lim_{n\to\infty} v_n = \max\{u_L, -\frac{1-\delta}{\delta}\}$, and is clearly uniquely defined.

Proposition 13. *Let $n \leq n^*$. Then*

$$w(p) = \begin{cases} p & \text{if } p \geq \overline{p}^{n^*+1} \\ p + \left(1 - \frac{p}{\overline{p}^{n^*}}\right)\delta\overline{w} - c(1-\delta) & \text{if } p < \overline{p}^{n^*+1} \end{cases}$$

[23]For notational simplicity we ignore the knife-edge cases where $v_n = 0$ for some n or $v_n = v_0^E$ for some n.

$$v(p) = \begin{cases} [v_n, v_{n-1})] & \text{if } p = \overline{p}^n \\ v_n & \text{if } p \in (\overline{p}^{n+1}, \overline{p}^n) \\ [0, v_{n^*)}] & \text{if } p = \overline{p}^{n^*+1} \\ 0 & \text{if } p < \overline{p}^{n^*+1} \end{cases}$$

and $v_0^E \in v(\overline{p}^{n^*})$.

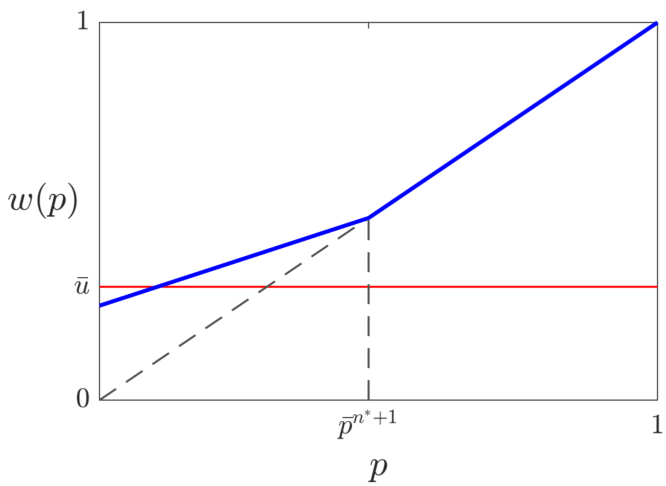

Figure 5: Principal's equilibrium value correspondence.

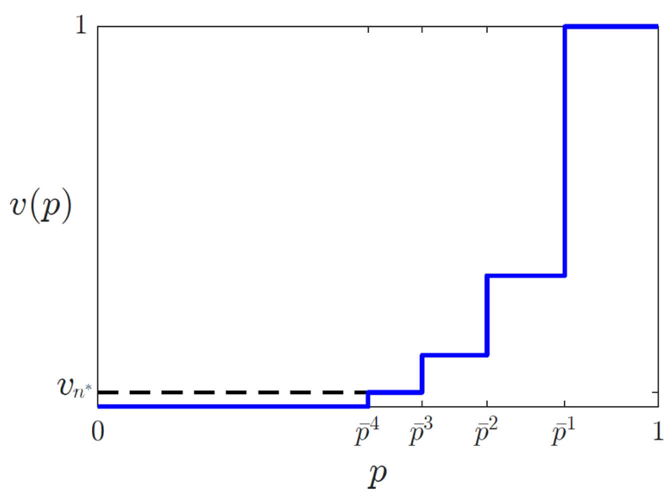

Figure 6: Agent's equilibrium value correspondence.

Proof. To simplify expressions, let $e := 1 - u_L$ and define $S_n := \sum_{i=0}^{n}(\delta e)^i$. Then we can write $v_n = \frac{1-e(1-\delta)S_{n-1}}{S_n}$. Now note that $\delta e S_n = S_{n+1} - 1$. I will use this a number of times throughout the proof. The following claim will be of use:
Subclaim: $v_0^E \in (v_{n^*}, v_{n^*-1})$.

Proof. Observe that $v_0^E > v_n$ implies that $v_{n+1} < 0$:

$$\frac{(1-\delta)(e-1)}{\delta} > \frac{1-e(1-\delta)S_{n-1}}{S_n}$$

iff

$$S_n(1+\delta+e-\delta e) > \delta - \delta e S_{n-1}(1-\delta)$$

iff

$$eS_n - S_{n+1} > 0$$

iff

$$1 - e(1-\delta)S_n < 0$$

which implies that $v_{n+1} < 0$. By an identical argument, $v_0^E < v_n$ implies that $v_{n+1} > 0$. Therefore since $v_{n^*} > 0 > v_{n^*+1}$, it must be that $v_{n^*} < v_0^E < v_{n^*-1}$. □

I proceed by induction to find the agent's equilibrium value correspondence for $p > \overline{p}^{n^*+1}$. Let $p = \overline{p}$. By claim 1, $m \leq e(1-\delta)$ and $p' = 1$ is necessary in equilibrium. Consider the stationary strategy by the principal which monitors with a fixed probability $m \leq e(1-\delta)$ every period until the first time he monitors. This is clearly an equilibrium with $v^E = w^E = 1$ and

$$v = v^N = \hat{v}(m) := \frac{(1-\delta)(1-m)}{1-\delta(1-m)}$$

defines the agent's equilibrium value as a function of m. Clearly $\hat{v}$ is continuous in m, and we have

$$1 = \hat{v}(0) \geq \hat{v}(m) \geq \hat{v}(e(1-\delta)) = v_1$$

for all $m \in [0, e(1-\delta)]$. By the intermediate value theorem, any $v \in [v_1, 1]$ is achievable in equilibrium, and clearly no payoff higher than 1 or lower than v_1 is feasible.
Now assume the induction hypothesis that for $1 \leq n \leq k < n^*$, $v(\overline{p}^n) = [v_n, v_{n-1}]$ and $v(p) = v_n$ for $p \in (\overline{p}^{n+1}, \overline{p}^n)$. Let $p = \overline{p}^{k+1} < \overline{p}$. Since $k < n^*$, by the subclaim, $v_k \geq v_{n^*-1} > v_0^E$. By the induction hypothesis, v_k is a lower bound on $v(p)$ for all $p > \overline{p}^{k+1}$. Therefore $v_0^E \notin v(p)$ for all $p > \overline{p}^{k+1}$, which implies that $0 \notin v(p)$, which implies that

$p' = \overline{p}^k$ is necessary in equilibrium. Since $p < \overline{p}$, $m = \frac{e(1-\delta)}{1-\delta+\delta v^E}$ must hold in equilibrium, with $v^E \in v(\overline{p}^k) = [v_k, v_{k-1}]$ by the induction hypothesis. Construct a stationary equilibrium which chooses the same v^E every period that the belief remains at p. This is an equilibrium which delivers

$$v = v^N = \tilde{v}(v^E) = \frac{(1-\delta)u_L + \delta m v^E}{1-\delta(1-m)}$$

Now $\tilde{v}$ is continuous in v^E and satisfies

$$\tilde{v}(v_{k-1}) \geq \tilde{v}(v^E) \geq \tilde{v}(v_k)$$

for all $v^E \in v(\overline{p}^k)$. By the intermediate value theorem, by varying v^E, every value in the interval is attainable in equilibrium, and clearly no other values are feasible. Therefore $v(\overline{p}^{k+1}) = [\tilde{v}(v_k), \tilde{v}(v_{k-1})]$. We need to show that $\tilde{v}(v_k) = v_{k+1}$:
Observe that $m = \frac{e(1-\delta)}{1-\delta+\delta v_k}$, or $m = e(1-\delta)S_k$. Then

$$\begin{aligned}\tilde{v}(v_k) &= \frac{(1-\delta)(1-e) + \delta e(1-\delta)S_k v_k}{1-\delta\big(1-e(1-\delta)S_k\big)} \\ &= \frac{S_{k+1} - eS_k}{S_{k+1}} \\ &= \frac{1-e(1-\delta)S_k}{S_{k+1}} = v_{k+1}\end{aligned}$$

which proves that $v(\overline{p}^{k+1}) = [v_{k+1}, v_k]$.
Let $p \in (\overline{p}^{k+2}, \overline{p}^{k+1})$. By the same argument as above, $v_0^E \notin v(p')$ for all $p' \geq \frac{p}{\overline{p}}$, which means that in equilibrium $p' = \frac{p}{\overline{p}}$, and by the induction hypothesis, $v^E = v_k$. This defines a unique equilibrium which delivers a payoff to the agent of $\tilde{v}(v_k) = v_{k+1}$. This completes the induction.
I have shown that for $p > \overline{p}^{n^*+1}$, equilibrium exists and for any equilibrium pair $(w, v) \in \mathcal{E}(p)$, $v > 0$. Thus by claim 2, $w = p$.
Let $p = \overline{p}^{n^*+1}$. By the subclaim, $v_0^E \in v(\overline{p}^{n^*})$, and by what has been shown above $v_0^E \notin v(p)$ for $p > \overline{p}^{n^*}$. Therefore, in equilibrium it is necessary that $p' = \overline{p}^{n^*}$, and $v^E \in [v_{n^*}, v_{n^*-1}]$. In the same way as before, we construct a stationary strategy for the principal. However, observe that this is only valid as long as $v^E \geq v_0^E$, since $\tilde{v}(v^E) < 0$ for $v^E < v_0^E$ and the agent cannot receive a negative payoff in equilibrium. Therefore $v^E \in [v_0^E, v_{n^*-1}]$. This defines an equilibrium, and by the same argument, we find that $v(\overline{p}^{n^*+1}) = [0, v_{n^*}]$. Now for any $(w, v) \in \mathcal{E}(p)$, in equilibrium we must have $p' = \frac{p}{\overline{p}}$, so $w = p$.

Lastly, let $p < \overline{p}^{n^*+1}$. If $p' = \frac{p}{\overline{p}} \in (\overline{p}^{n^*+1}, \overline{p}^{n^*})$ is possible in equilibrium, then since $v(p') = v_{n^*}$, it delivers the agent a payoff of

$$\tilde{v}(v_{n^*}) = v_{n^*-1} < 0$$

a contradiction. Therefore, it must be that $p' > \frac{p}{\overline{p}}$ which requires that $v^E = v_0^E$. By subclaim $v_0^E \in (v_{n^*}, v_{n^*-1}$ and by our construction of $v(p)$, $v_0^E \notin v(p)$ for all other p. So it must be that $p' = \overline{p}^{n^*}$. This defines an equilibrium with $v = 0$. Furthermore, the principal's value in this equilibrium is

$$\begin{aligned} w &= p + s_L(1-p)\delta\overline{w} - c(1-\delta) \\ &= p + \left(1 - \frac{p}{\overline{p}^{n^*}}\right)\delta\overline{w} - c(1-\delta) \end{aligned}$$

and no other equilibrium payoff is feasible at such p. □

The proof of theorem 1 is now a simple application of the preceding results. I assume that $p_0 \neq \overline{p}^n$ for all n. Since this set of priors is a set of Lebesgue measure zero, all results are generically true.

Proof of Theorem 1 and Proposition 1. Existence of equilibrium comes from the proof of proposition 4 , so let σ be an equilibrium. Suppose $p_0 > \overline{p}^{n^*+1}$, and $p_0 \in (\overline{p}^{n+1}, \overline{p}^n)$. By proposition 4, $V(\sigma) = v(p_0) = v_n$ and $W(\sigma) = p_0$. Now $v_0^E \notin v(p')$ for all $p' \geq \frac{p_0}{\overline{p}}$, which implies that for all h such that $p(h) < \overline{p}$, $p(hE) = \frac{p(h)}{\overline{p}}$. Thus p does not hit $\overline{p}$ in equilibrium. For all h such that $p(h) > \overline{p}$, $p(hE) = 1$. Therefore the agent's strategy is uniquely defined and Markovian with respect to the p. Given this strategy, for h such that $p(h) < \overline{p}$, $p(h) \in (\overline{p}^{k+1}, \overline{p}^k)$ for some $k \leq n$. This implies that

$$m(h) = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E} = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v_k} = (1-\delta)(1-u_L)\sum_{i=0}^{k}\delta^i(1-u_L)^i$$

For h such that $p(h) > \overline{p}$, $m(h) = 0$. This shows that m is uniquely defined and Markovian with respect to p.
Suppose $p_0 < \overline{p}^{n^*+1}$. By proposition 4,

$$W(\sigma) = w(p_0) = p_0 + \left(1 - \frac{p_0}{\overline{p}^{n^*}}\right)\delta\overline{w} - c(1-\delta)$$

and $V(\sigma) = v(p_0) = 0$. By the proof of proposition 4, it must be that $p(\emptyset M) = \overline{p}^{n^*}$. Now

$v_0^E \notin v(p')$ for all $p' > \overline{p}^{n^*}$, which implies that $V(\sigma)|_h > 0$ for all h after the null history. Therefore $p(hE) = \frac{p(h)}{\overline{p}}$ for all such h. Thus the agent's equilibrium strategy is uniquely determined and Markovian with respect to p.

The proof of proposition 1 follows from the agent's equilibrium strategy. Let (σ) be an equilibrium. For $p_0 > \overline{p}^{n^*+1}$, $V(\sigma) = v(p_0) > 0$, so the sequence of beliefs must satisfy $p' = \frac{p}{\overline{p}}$ until the belief crosses $\overline{p}$. For $p_0 < \overline{p}^{n^*+1}$, $V(\sigma) = v(p_0) = 0$, so the initial jump in the belief must be from p_0 to $\overline{p}^{n^*}$, where $v_0^E > 0$ is delivered, after which the sequence satisfies $p' = \frac{p}{\overline{p}}$ until the belief crosses $\overline{p}$. □

8 Appendix B: Adverse Selection and Moral Hazard

8.1 Optimal Contract

Proof of existence of optimal contract in Theorem 3: I omit the proof since it is standard and analagous to the proof in Theorem 1.

Proof of proposition 8:

Proof. The proof is analogous to that of proposition 1 in the pure adverse selection case. □

Auxiliary Problem: Pareto Frontier

Consider the auxiliary problem of a contract between the principal and a known high type of agent. For this section I will refer to the high type as the agent. Let $\sigma^H = (d, m, s_H)$ be contract between the principal and the agent.

Define $\mathcal{E}_H$ be the set of all pairs of principal and agent values possible through a contract:

$$\mathcal{E}_H := \{ \left(W(\sigma^H), V_H(\sigma^H)\right) | \sigma^H \text{is a contract}\}$$

The recursive formulation thus specifies actions and continuation values after the respective public histories for each player:

Definition 7 (Contract values)**.** *Let $(w, v) \in \mathcal{E}_H$. Then there exist actions $d, m \in [0, 1]$, $s_H \in \{0, 1\}$, and continuation values $(w^N, v^N), (w^E, v^E), (w^S, v^S) \in \mathcal{E}_H$ such that promise-keeping, optimality of the principal's firing decision, and the agent's incentive constraint* *holds:*

$$w = (1-d)\bar{w} + d\big[(1-\delta)(1-s_H-mc) + \delta(1-m)w^N + \delta m\left((1-s_H)w^E + s_H w^S\right)\big] \quad \text{(PPK)}$$

$$v = d\big[(1-\delta)\big((1-s_H)u_H + s_H(1-m)\big) + \delta(1-m)v^N + \delta m\left((1-s_H)v^E + s_H v^S\right)\big] \quad \text{(APK)}$$

$$d \in \underset{d' \in [0,1]}{\arg\max}\ (1-d')\bar{w} + d'\big[(1-\delta)(1-s_H-mc) + \delta(1-m)w^N + \delta m\big((1-s_H)w^E + s_H w^S\big)\big] \quad \text{(PFD)}$$

$$s_H \in \underset{s' \in \{0,1\}}{\arg\max}\ (1-\delta)\big((1-s')u_H + s'(1-m)\big) + \delta(1-m)v^N + \delta m\big((1-s')v^E + s'v^S\big) \quad \text{(AIC)}$$

By APS, $\mathcal{E}_H$ is compact. Define $\bar{v}$ as the maximum value (which exists by compactness) the agent can receive from a contract:

$$\bar{v} := \max\{v|(w,v) \in \mathcal{E}_H\}$$

Observe that there exists δ^* such that if $\delta > \delta^*$ then $\bar{v} > u_H$: there exists a contract in which the agent always exerts effort and the principal always monitors, in which if anybody deviates, the agent always shirks from then on and the principal always fires. This is incentive compatible for the agent by the assumption that $\delta > 1 - u_H$. Therefore the contract which allows the agent to shirk in the first period with no monitoring, followed by the contract just mentioned gives the agent a value of $(1-\delta) + \delta u_H > u_H$. The principal's payoff from this is $\delta(1-c)$. Therefore if $\delta > \frac{\bar{w}}{1-c}$ this contract is feasible.

Define the Pareto Frontier of $\mathcal{E}_H$ as:

$$PF(\mathcal{E}_H) := \{(w,v) \in \mathcal{E}_H |\ \nexists (w',v') \in \mathcal{E}_H \text{ s.t. } (w',v') > (w,v)\}$$

Define as the minimum agent value on the Pareto Frontier,

$$v^* := \min\{v|(w,v) \in PF(\mathcal{E}_H)\}$$

Clearly, if $(w^*, v^*) \in PF(\mathcal{E}_H)$ then w^* is the principal's best payoff from a contract.

Define the function which takes agent values on the Pareto Frontier to the corresponding principal value on the Pareto Frontier as:

$$F^P(v) := \{w|(w,v) \in PF(\mathcal{E}_H)\}$$

Then clearly the domain of F^P is $\{v|(w,v) \in PF(\mathcal{E}_H)\}$. For every v in this domain, $F^P(v)$

is the maximum payoff the principal can get from a contract given that the agent is getting v.

I now define the set of values generated by contracts with no on-path firing. It will turn out that this set will contain the Pareto Frontier.

$$\mathcal{E}_H^1 := \{\big(W(\sigma^H), V_H(\sigma^H)\big) \mid \sigma^H \text{is a contract s.t. } \mathbb{P}^{\sigma^H}(h^t) > 0 \implies d(h^t) = 1\}$$

Claim 7. $PF(\mathcal{E}_H) \subset \mathcal{E}_H^1$.

Proof. Let $(w, v) \in PF(\mathcal{E}_H)$. Then it must be that $d = 1$. To see this, if $d < 1$, then $w = \bar{w}$, since otherwise the principal would not fire. Clearly $v < \bar{v}$. Therefore $(\bar{w}, \bar{v}) > (w, v)$, contradicting that $(w, v) \in PF(\mathcal{E}_H)$.
Now suppose $(w^N, v^N) \notin PF(\mathcal{E}_H)$ and N is on-path. Then there exists $(w', v') > (w^N, v^N)$. Therefore the contract which replaces (w^N, v^N) with (w', v') and keeps everything else fixed delivers a strictly higher payoff for both players and is incentive compatible, contradicting that $(w, v) \in PF(\mathcal{E}_H)$.
Suppose $(w^E, v^E) \notin PF(\mathcal{E}_H)$ and E is on-path. Then it must be that $s_H = 1$, so the agent's incentive constraint requires that

$$m \geq \frac{(1-\delta)(1-u_H)}{1-\delta+\delta(v^E - v^S)}$$

There exists $(w', v') > (w^E, v^E)$. Replacing (w^E, v^E) with $(w'v')$ and keeping everything else fixed preserves the incentive constraint for the agent and increases both players' payoffs, a contradiction. An analogous argument follows for the case $(w^S, v^S) \notin PF(\mathcal{E}_H)$.
This shows that if we are on the Pareto Frontier, there is not firing today, and since all on-path continuation values stay on the Pareto Frontier, there is no firing on-path for contracts on the Pareto Frontier. $\square$

Note that for contracts that generate values in $\mathcal{E}_H^1$, the agent's payoff is always weakly greater than u_H, and includes u_H. I therefore define the function

$$H(v) := \max\{w | (w, v) \in \mathcal{E}_H^1\}$$

It is straightforward to see that the domain of H is $[u_H, \bar{v}]$ since we can construct a contract which delivers every value in this interval (the construction will be evident from the proof) and since $\mathcal{E}_H^1$ is compact.

H is a fixed point of the operator $T : \mathcal{B}[u_H, \bar{v}] \to \mathcal{B}[u_H, \bar{v}]$:

$$
\begin{aligned}
Tf(v) = \max_{d,m,s_H,v^N,v^E,v^S} & (1-d)\bar{w} + d\Big[(1-\delta)(1-s_H-mc) + \delta(1-m)f(v^N) \\
& + \delta m \left((1-s_H)f(v^E) + s_H f(v^S)\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.} \quad v =& d\big[(1-\delta)((1-s_H)u_H + s_H) + \delta\left((1-m)v^N + m((1-s_H)v^E + s_H v^S)\right)\big] \qquad (PK) \\
s_H \in& \arg\max_{\tilde{s}_H \in \{0,1\}} \ (1-\delta)((1-\tilde{s}_H)u_H + \tilde{s}_H) + \delta\left((1-m)v^N + m((1-\tilde{s}_H)v^E + \tilde{s}_H v^S)\right) \qquad (IC) \\
0 \leq& m s_H (v^S - u_H) \\
0 \leq& m(1-s_H)(V^E - u_H) \\
0 \leq& v^N - u_H
\end{aligned}
$$

where the last three constraints are the requirement that any on-path continuation values are above u_H, and $v^N \geq u_H$ is without loss even when N is off-path since v^N does not affect incentives. A standard application of Blackwell's sufficient conditions for a contraction show that T is a contraction under the sup-norm, so by Banach's Fixed Point Theorem, must have a unique fixed point.

I now propose a candidate for the solution, $\hat{H}$.

Definition 8. *Define the sequence $\{v_k\}_{k=0}^{\infty}$: let $v_0 : \bar{v}$ and let v_k satisfy the recursion setting $v^N = v$ in PK and IC binding:*

$$v_k = (1-m)(1-\delta+\delta v_k)$$

and

$$m = \frac{(1-\delta)(1-u_H)}{1-\delta+\delta v_{k-1}}$$

The solution to this recursion is given below and gives a decreasing sequence which converges to u_H.

Observation 4.

$$v_k = \frac{1-\delta(1-\bar{v}) - (1-u_H)^k \delta^{k-1}(1-\delta) - (1-\delta)(1-\delta(1-\bar{v}))\sum_{i=1}^{k-1}\delta^{i-1}(1-u_H)^i}{1-(u_H-\bar{v})\sum_{i=1}^{k}\delta^i(1-u_H)^{i-1}}$$

Definition 9. *Define the sequence of functions* $\{\hat{H}_k\}_{k=1}^{\infty}$ *where* $\hat{H}_k : [u_h, \bar{v}]$*: let* $\hat{H}_1(v) = \frac{1-v}{1-\bar{v}}\bar{w}$ *and let* $\hat{H}_k$ *satisfy the recursion*

$$\hat{H}_k(v) = \frac{(1-\delta)(1-mc) + \delta m \hat{H}_{k-1}(v^E)}{1-\delta(1-m)}$$

such that

$$v = (1-m)[1-\delta+\delta v]$$

and

$$m = \frac{(1-\delta)(1-u_H)}{1-\delta+\delta v^E}$$

The function $\hat{H}_1$ is an upper bound on the principal's payoff from delivering any v to the high type. We guess that for v sufficiently high, this upper bound is feasible. The recursion is defined by guessing that in general it will be optimal to have the high type exert effort with the incentive constraint binding, and setting $v^N = v$. Furthermore, it will be optimal to set $v^S = 0$ as the harshest off-path punishment. This gives us the two constraints in the definition of the recursion. The solution to the recursion is given below:

Observation 5.

$$\begin{aligned}\hat{H}_k(v) = v & \left(\delta \sum_{i=0}^{n-2} \delta^i (1-u_H)^i + c(1-\delta)\sum_{i=0}^{n-2}(n-1-i)\delta^i(1-u_H)^i - \frac{\bar{w}}{1-\bar{v}}\sum_{i=0}^{n-1}\delta^i(1-u_H)^i\right) \\ & + (1-\delta)\left(\sum_{i=0}^{n-2}\delta^i(1-u_H)^i - c\left(n-1-(1-\delta)(1-u_H)\sum_{i=0}^{n-3}(n-2-i)\delta^i(1-u_H)^i\right)\right) \\ & + \frac{\bar{w}}{1-\bar{v}}\left(u_H + \delta(1-u_H) - (1-\delta)(1-u_H)\sum_{i=0}^{n-3}\delta^{i+1}(1-u_H)^{i+1}\right)\end{aligned}$$

Define the function $\hat{H} : [u_H, \bar{v}] \to [0,1]$ as

$$\hat{H}(v) := \hat{H}_k(v)$$

for $v \in [v_k, v_{k-1})$.

Observation 6. *(Properties of* $\hat{H}$*)*
i) $\hat{H}$ *is piecewise linear and concave. ii) There exists* n *such that* $\hat{H}$ *is strictly decreasing (generically) for* $v \geq v_n$ *and strictly (generically) increasing for* $v < v_n$*.*

Proof. i) readily follows from the definition of $\hat{H}$ and by noting that

$$\hat{H} = \min_{k \in \mathbb{N}} \hat{H}_k$$

where $\hat{H}_K$ is the linear function $\hat{H}$ is defined as for $v \in [v_k, v_{k-1})$.
ii) follows by noting that $\hat{H}_1' < 0$ and $\hat{H}_k' \to \infty$ as $k \to \infty$. □

We wish to show that $\hat{H}$ is the fixed point of T.

The following lemmas prove properties of the operator which reduce it to a more tractable form. I now prove some properties of the T when applying it to $\hat{H}$.

Let $\gamma = (d, m, s_H, v^N, v^S, v^E)$ denote the policy today. The first lemma states that the principal monitors if and only if the agent exerts effort today.

Lemma 4. *Let γ be an optimal policy at $T\hat{H}(v)$. Then $m > 0$ iff $s_H = 0$.*

Proof. Clearly if $m = 0$ then it must be that $s_H = 1$ since the payoff from shirking is strictly higher than effort:

$$(1-\delta) + \delta v^N > (1-\delta)u_H + \delta v^N$$

which proves the if direction.
For the only if, suppose for the sake of contradiction that $s_H = 1$ and $m > 0$. Then

$$\big(\hat{H}(v), v\big) = (1-\delta)(1-m)(0,1) + (1-\delta)m(-c,0) + \delta(1-m)\big(\hat{H}(v^N), v^N\big) + \delta m\big(\hat{H}(v^S), v^S\big)$$

There are 3 possible cases:
Case 1: $v^S \leq v \leq v^N$ or $v^N \leq v \leq v^S$.
Define $L(v) := \alpha\hat{H}(v^N) + (1-\alpha)\hat{H}(v^S)$, where α is such that $v = \alpha v^N + (1-\alpha)v^S$. By concavity of $\hat{H}$, $\hat{H}(v) \geq L(v)$.
Since $m > 0$, $\big(\hat{H}(v), v\big) \in int\Big\{co\{(0,1), (-c,0), \big(\hat{H}(v^N), v^N\big), \big(\hat{H}(v^S), v^S\big)\}\Big\}$, a contradiction.
Case 2: $v^S, v^N \leq v$. Suppose $v \leq v^*$. $\hat{H}$ is increasing in the range $[u_H, v^*]$ by concavity, so $F^H(v^N) \leq \max\Big\{\hat{H}(v), \hat{H}(v^N)\Big\}$. But

$$\hat{H}(v) = -mc(1-\delta) + \delta(1-m)\hat{H}(v^N) + \delta m\hat{H}(v^S) < max\{\hat{H}(v^N), \hat{H}(v^S)\}$$

a contradiction.
Suppose $v \geq v^*$. Then by concavity, $\hat{H}$ is decreasing on $[v^*, \bar{v}]$. By the argument in the

proof of Claim 7, on-path continuation values are on the Pareto Frontier, so $v^N, v^S \geq v^*$. Construct the alternative policy $m' = 0$ $s'_H = 1$, $v^{N'} = (1-m)v^N + mv^S - m\frac{1-\delta}{\delta}$. This is clearly incentive compatible and PK holds since

$$1 - \delta + \delta v^{N'} = (1-m)(1-\delta+\delta v^N) + \delta m v^S = v$$

where the last equality follows from the fact that the original policy satisfied PK. The principal's payoff from this policy is

$$\begin{aligned}\hat{H}\big((1-m)v^N + mv^S - m\frac{1-\delta}{\delta}\big) &\geq \hat{H}\big((1-m)v^N + mv^S\big) \\ &> -mc(1-\delta) + \delta(1-m)\hat{H}(v^N) + m\hat{H}(v^S)\end{aligned}$$

where the first inequality is because $\hat{H}$ is increasing in this range and the second is by concavity of $\hat{H}$. This is an improvement, contradicting the optimality of the policy γ. Case 3: $v^S, v^N \geq v$. Then PK fails:

$$(1-\delta)(1-m) + \delta(1-m) + \delta m v^S > v$$

a contradiction. □

Notice that this lemma implies that shirking is never observed by the principal on-path. As a result, we can set the continuation value to the agent after the event that shirking is observed to 0, the harshest punishment.

Lemma 5. *Let γ be an optimal policy at $T\hat{H}(v)$. Then wlog $v^S = 0$.*

Proof. Suppose $v^S > 0$. By lemma 4, if $s_H = 1$ then $m = 0$, so v^S is off-path, and shirking is a strict best response independent of v^S. Similarly, if $s_H = 0$, v^S is off-path. The agent's incentive constraint to work requires

$$m \geq \frac{(1-\delta)(1-u_H)}{1-\delta+\delta(v^E - v^S)}$$

Therefore setting $v^S = 0$ does not change incentives so it is without loss to do so. □

The next lemma says the agent's IC must bind whenever the agent is exerting effort.

Lemma 6. *Let γ be an optimal policy at $T\hat{H}(v)$ such that $s_H = 0$. Then the agent's IC binds.*

Proof. Suppose for the sake of contradiction that the agent's IC is slack, so:

$$m > \frac{(1-\delta)(1-u_H)}{1-\delta+\delta v^E}$$

Then define the alternative policy γ': $s'_H = 0$, $m' = \frac{(1-\delta)(1-u_H)}{1-\delta+\delta v^E}$, $v^{N'} = \frac{v-(1-\delta)u_H-\delta m' v^E}{\delta(1-m')}$, $v^{E'} = v^E$.

Clearly $m' < m$ so the new policy satisfies incentives for the agent to exert effort. PK also holds since the agent's payoff from effort is

$$(1-\delta)u_H + \delta(1-m')v^{N'} + \delta m' v^E = (1-\delta)u_H + v - (1-\delta)u_H - \delta m' v^E + \delta m' v^E = v$$

Furthermore, the new policy is feasible since by PK,

$$v^{N'} = \frac{v-(1-\delta)u_H-\delta m' v^E}{\delta(1-m)} > \frac{v-(1-\delta)u_H-\delta m v^E}{\delta(1-m)} = v^N \geq u_H$$

and $v^E \geq u_H$.

To see that the new policy is a strict improvement for the principal, define $\lambda = \frac{1-m}{1-m'} \in (0,1)$. By PK, $v^N = \frac{v-(1-\delta)u_H-\delta m v^E}{\delta(1-m)}$ so

$$\begin{aligned}
\lambda v^N + (1-\lambda)v^E &= \frac{1-m}{1-m'}\frac{v-(1-\delta)u_H-\delta m v^E}{\delta(1-m)} + \frac{m-m'}{1-m'}v^E \\
&= \frac{1}{\delta(1-m')}\left(v-(1-\delta)u_H-\delta m v^E + \delta m v^E - \delta m' v^E\right) \\
&= v^{N'}
\end{aligned}$$

The principal's payoff gain from the new policy is

$$(1-\delta)(m-m')c + \delta\left((1-m')f(v^{N'}) + m' f(v^E) - (1-m)f(v^N) - m f(v^E)\right)$$

which equals

$$(1-\delta)(m-m')c + \delta(1-m')\left(f(v^{N'}) - \left(\frac{1-m}{1-m'}f(v^N) + \frac{m-m'}{1-m'}f(v^E)\right)\right)$$

which is the same as

$$(1-\delta)(m-m')c+\delta(1-m')\Bigg(f\big(\lambda v^N+(1-\lambda)v^E\big)-\Big(\lambda f(v^N)+(1-\lambda)f(v^E)\Big)\Bigg)>0$$

where the inequality follows from the fact that $m' < m$ and the concavity of f. $\square$

As a result of this lemma, we can rewrite the agents PK constraint as if the agent were shirking all the time:

$$v = 1-\delta+\delta(1-m)v^N$$

By lemma 4 the IC can be rewritten as

$$m=\frac{(1-\delta)(1-u_H)}{\delta v^E}(1-s_H)$$

since $m = 0$ if the agent shirks, but the IC binds if the agent exerts effort.
Furthermore, we can assume that continuation values v^N and v^E are chosen in $[u_H, \bar{v}]$ without loss since if these are off-path they don't affect any incentives. This simplifies the operator T to:

$$Tf(v) = \max_{\substack{m\in[0,1],s_H\in\{0,1\},\\ v^N,v^E\in[u_H,\bar{v}]}} (1-\delta)(1-s_H-mc)+\delta\left((1-m)f(v^N)+m\left((1-s_H)f(v^E)\right)\right)$$

$$\text{subject to} \quad v=(1-m)(1-\delta+\delta v^N) \qquad (PK)$$

$$m=\frac{(1-\delta)(1-u_H)}{\delta v^E}(1-s_H) \qquad (IC)$$

I now show that $\hat{H}$ is indeed the fixed point of the operator.

Claim 8. *There exists $\delta^* \in (0,1)$ such that if $\delta > \delta^*$ then $T\hat{H} = \hat{H}$.*

Proof. For $v \in [v_1, \bar{v}]$ I will show that $H(v) = \frac{1-v}{1-\bar{v}}\bar{w} = \hat{H}(v)$. First, note that $H(\bar{v}) = \bar{w}$. This follows from the continuity of H due the compactness of the set $\mathcal{E}_H^1$. Suppose for the sake of contradiction that $H(\bar{v}) > \bar{w}$. Since $\bar{v} > (1-\delta)u_H + \delta\bar{v}$, the agent must be shirking today. Therefore by lemma 4 the principal does not monitor, so the principal's payoff is

$$\delta H\left(\frac{\bar{v}-(1-\delta)}{\delta}\right) > \bar{w}$$

Construct a new contract such that $m = 0$, $s_H = 1$, $v^N = \frac{\bar{v}+\epsilon-(1-\delta)}{\delta}$. The principal's payoff from this contract, for ϵ sufficiently small, is

$$\delta H\left(\frac{\bar{v}+\epsilon-(1-\delta)}{\delta}\right) > \bar{w}$$

so this is a contract, and the agent's payoff is $\bar{v}+\epsilon > \bar{v}$ contradicting that $\bar{v}$ is the highest payoff the agent can get from a contract. Therefore $H(\bar{v}) = \bar{w}$.

Now observe that in order to have the agent exert effort today, the principal must monitor with some minimum probability, since the agent exerts effort only if

$$m \geq \frac{(1-\delta)(1-u_H)}{1-\delta+\delta v^E} \geq \frac{(1-\delta)(1-u_H)}{1-\delta+\delta\bar{v}} = \bar{m}$$

Therefore the maximum feasible payoff in any period for the principal is $\bar{y} := 1-\bar{m}c$, and when the principal gets this, the agent gets u_H. Similarly, the maximum feasible payoff for the agent in a period is 1, and if he shirks, we know the principal gets 0 since he does not monitor. Therefore the line joining the value pairs $(\bar{y}, u_H)$ and $(0, 1)$ is an upper bound on the set of contract payoffs.

The function $\hat{H}_1(v)$ is in fact the projection of principal values onto this line, so we have to show that this upper bound is achievable with a contract (for δ sufficiently high). I will show that the set

$$L := \left\{\left(v, \frac{1-v}{1-\bar{v}}\bar{w}\right) \mid v \in v[v_1, \bar{v}]\right\}$$

is self-generating. Define first $\tilde{v} := (1-\delta)u_H + \delta\bar{v}$. This is the highest value such that it is still possible to have the agent exert effort. Above this having the agent exert effort will violate promise keeping. If indeed, the set is self-generating then the point $(\bar{w}, \bar{v})$ lies on the line joining $(0, 1)$ and $(\bar{y}, u_H)$. Therefore $\bar{v}$ satisfies

$$\bar{w} = \frac{1-\bar{v}}{1-u_H}\bar{y}$$

which gives us a quadratic equation in terms of $\bar{v}$ with one solution above u_H since $\delta > 1-u_H$. $\delta > 1-u_H$ also implies that $\tilde{v} > v_1$. Therefore $\tilde{v} \in (v_1, \bar{v})$. Using the solution for $\bar{v}$, it is easy verified that there is a cutoff δ^* such that if $\delta > \delta^*$ then

$$\frac{v-(1-\delta)}{\delta} > v_1$$

for any $v \in (v_0, \bar{v}]$.
Now let $(w, v) \in L$ and $v \in (v_0, \bar{v}]$. Then it must be that the agent is shirking and today. Define the policy as $s_H = 1$, $m = 0$, $v^N = \frac{v-(1-\delta)}{\delta}$. Then clearly agent promise keeping is satisfied, and if the continuation payoffs (w^N, v^N) lie in L, the principal's payoff is

$$\delta \frac{1 - \frac{v-(1-\delta)}{\delta}}{1-\bar{v}} \bar{w} = \frac{1-v}{1-\bar{v}} \bar{w}$$

For $\delta > \delta^*$, $v^N > v_1$ so this is feasible with continuation payoffs from L. Now let $(w, v) \in L$ and $v \in [v_1, \tilde{v}]$. Define the policy $s_H = 0$, $v^E = \bar{v}$ and $v^S = 0$ and v^N and m such that PK holds and IC binds. Then by definition of v_1, $v^N \in [v_1, \bar{v}]$, so the principal's payoff is

$$(1-\delta)(1-mc) + \delta(1-m)\frac{1-v^N}{1-\bar{v}}\bar{w} + \delta m \bar{w} = \frac{1-v}{1-\bar{v}}\bar{w}$$

Therefore the set L is self-generating and the upper bound on the set of payoffs is achieved. Therefore for $v \in [v_1, \bar{v}]$,

$$H(v) = \frac{1-v}{1-\bar{v}}\bar{w} = \hat{H}(v)$$

so it must be that $T\hat{H}(v) = H(v)$.
Let $v \in [v_k, v_{k-1})$ for $k > 1$. Since the slope of $\hat{H}$ is increasing as we move down intervals, $\delta \hat{H}(\frac{v-(1-\delta)}{\delta}) < \hat{H}(v)$. Therefore it must be that the agent is exerting effort. For any policy γ, define

$$J(\gamma) := (1-\delta)(1-mc) + \delta(1-m)\hat{H}(v^N) + \delta m \hat{H}(v^E)$$

and

$$J_k(\gamma) := (1-\delta)(1-mc) + \delta(1-m)\hat{H}_k(v^N) + \delta m \hat{H}_{k-1}(v^E)$$

Then $J(\gamma) \leq J_k(\gamma)$. Define the relaxed problem

$$\begin{aligned} H_k^*(v) = &\max_{\gamma} \; J_k(\gamma) \\ \text{subject to} \quad & v = (1-m)[1-\delta+\delta v^N] && (PK) \\ & m = \frac{(1-\delta)(1-u_H)}{1-\delta+\delta v^E} && (IC) \end{aligned}$$

Clearly $T\hat{H}(v) \leq H_k^*(v)$. Now for any policy γ such that PK holds and IC binds,

$$J_k(\gamma) = \hat{H}_k(v)$$

Therefore if we can find a policy such that $v^N \in [v_k, v_{k-1}]$, $v^E \in [v_{k-1}, v_{k-2}]$, PK holds and IC binds then $T\hat{H}(v) = \hat{H}_k(v) = \hat{H}(v)$.
Construct the policy such that $v^N = v$, and let PK hold and IC bind. By lemma ? then $v^E \in [v_{k-1}, v_{k-2}]$ so we are done. We have shown that $T\hat{H} = \hat{H}$. □

Therefore $H(v) = \hat{H}(v)$. Having solved for this we can see what F^P must be. By definition of the Pareto Frontier, F^P must be a decreasing function. H is first increasing then decreasing on $[u_H, \bar{v}]$, and by observation 6 there exists v_n such that H is decreasing for $v \geq v_n$. Define

$$n := \left\{k | H'_{k+1} > 0 > H'_k\right\}$$

and let v_n be defined accordingly. Then it is evident that

Proposition 14. $F^P : [v_n, \bar{v}]$ *and*

$$F^P = H|_[v_n, \bar{v}]$$

.

Phase 2: Optimal Delivery

I now return original problem or optimally delivering v to the low type.

For the recursive formulation of the value delivery problem, define the set of feasible triples of payoffs from contracts for the principal and the two types of agent as

$$\mathcal{E} := \{(W(\sigma), V_H(\sigma), V_L(\sigma)) \mid \sigma \text{ a contract, } p_0 = 1\}$$

A triple of values in $\mathcal{E}$ is generated by a firing decision, monitoring probability and shirking decisions for each type, and continuation values for the principal and each type of agent after each public event, which must come from $\mathcal{E}$ themselves.

Definition 10. *(Contract Values). Let $(w, v_H, v_L) \in \mathcal{E}$. Then there exist actions $d, m \in [0, 1]$, $s_L, s_H \in \{0, 1\}$, and continuation values $(w^N, v_H^N, v_L^N), (w^E, v_H^E, v_L^E), (w^S, v_H^S, v_L^S) \in \mathcal{E}$ such that promise-keeping holds for each player, and the principal's firing decision and both type of agents' actions are optimal:*

$$w = (1-d)\bar{w} + d\Big[(1-\delta)(1-s_H-mc) + \delta(1-m)w^N + \delta m\left((1-s_H)w^E + s_H w^S\right)\Big] \quad \text{(PPK)}$$

$$v_H = (1-\delta)\big((1-s_H)u_H + s_H(1-m)\big) + \delta(1-m)v_H^N + \delta m\left((1-s_H)v_H^E + s_H v_H^S\right) \quad \text{(HPK)}$$

$$v_L = (1-\delta)\big((1-s_L)u_L + s_L(1-m)\big) + \delta(1-m)v_L^N + \delta m\left((1-s_L)v_L^E + s_L v_L^S\right) \quad \text{(LPK)}$$

$$d \in \arg\max_{d'\in[0,1]} \; (1-d')\bar{w} + d'\Big[(1-\delta)(1-s_H-mc) + \delta(1-m)w^N + \delta m((1-s_H)w^E + s_H w^S)\Big] \quad \text{(PFD)}$$

$$s_H \in \arg\max_{s'\in\{0,1\}} \; (1-\delta)\big((1-s')u_H + s'(1-m)\big) + \delta(1-m)v_H^N + \delta m\left((1-s')v_H^E + s'v_H^S\right) \quad \text{(AIC)}$$

$$s_L \in \arg\max_{s'\in\{0,1\}} \; (1-\delta)\big((1-s')u_L + s'(1-m)\big) + \delta(1-m)v_L^N + \delta m\left((1-s')v_L^E + s'v_L^S\right) \quad \text{(AIC)}$$

Remark 2. *If* $(w, v_H, v_L) \in \mathcal{E}$, *then* $v_H \geq v_L$.

This is immediate from the fact that the high type has a higher payoff when exerting effort - he can always mimic the low type and get a weakly better payoff.

Formally, $F(v)$ is defined as

$$F(v) := \max\{w | (w, v_H, v) \in \mathcal{E}\}$$

For any value v define the set of values $v_H(v)$ as the set of values that can be delivered to the high type if v is being delivered to the low type:

$$v_H(v) := \{v_H | (w, v_H, v) \in \mathcal{E}\}$$

We have to choose values from this set whenever we are delivering some value v to the low type. Define a policy $\gamma := (d, m, s_H, s_L, v^N, v^E, v^S, v_H^E, v_H^S)$ s.t. $v^N, v^E, v^S \in [0,1]$, $v_H^E \in v_H(v^E)$, $v_H^S \in v_H(v^S)$. F is then the fixed point of the following operator:

$$\begin{aligned} Tf(v) = \max_{\gamma} \quad & (1-d)\bar{w} \\ & + d\Big[(1-\delta)(1-s_H-mc) + \delta(1-m)f(v^N) + dm\left[(1-s_H)f(v^E) + s_H f(v^S)\right]\Big] \end{aligned}$$

subject to

$$v = d\Big[(1-\delta)\left[(1-s_L)u_L + s_L(1-m)\right] + \delta(1-m)v^N + \delta m\left[(1-s_L)v^E + s_L v^S\right]\Big] \qquad (PK)$$

$$s_H \in \arg\max_{s'}(1-\delta)\left[(1-s')u_H + s'(1-m)\right] + \delta(1-m)v^N + \delta m\left[(1-s')v_H^E + s'v_H^S\right] \qquad (ICH)$$

$$s_L \in \arg\max_{s'}(1-\delta)\left[(1-s')u_L + s'(1-m)\right] + \delta(1-m)v^N + \delta m\left[(1-s')v^E + s'v^S\right] \qquad (ICH)$$

Having solved the auxiliary problem, it is straightforward to show that for $v \in [v_n, \bar{v}]$ the solution to the auxiliary problem coincides with the solution to the overall problem:

Claim 9. *Let $v \in [v_n, \bar{v}]$. Then $F(v) = F^P(v)$.*

Proof. First we show that $F(v) \le F^P(v)$. Let $(w, v_H, v) \in \mathcal{E}$. Then it must be that $v_H \ge v$. This implies that $w \le F^P(v)$ since if $v_H \ge v$, the principal's payoff is at most $F^P(v)$ in the auxiliary problem which is less constrained than the main problem. Therefore $F(v) \le F^P(v)$. Now take the policy which delivers $\big(F^P(v), v\big)$ in the auxiliary problem. This involves continuation values for the high type in $[v_n, \bar{v}]$. Set the low type's action to $s_L = 1$ at every history. This is incentive compatible since the high type always weakly prefers to shirk - he is indifferent when exerting effort. Furthermore, since the high type's payoff from shirking everywhere is v, so is the low type's. Therefore this policy delivers v to the low type and gives the principal a payoff of $F^P(v)$. Therefore $F(v) = F^P(v)$. □

I now propose a candidate for for the solution, $\hat{F}$. Since we know the function above v_n, it remains to construct the candidate for $v < v_n$.

Definition 11. *Let $\{v_k\}_{k=0}^n$ be as given in the auxiliary problem. Define the continued sequence $\{v_k\}_{k=n+1}^{\infty}$ as: v_{n+1} satisfies*

$$v_{n+1} = (1-m)[1-\delta+\delta v_{n+1}]$$

and

$$m = \frac{(1-\delta)(1-u_H)}{1-\delta+\delta v_n}$$

and for $k \ge n+2$, v_k satisfies

$$v_k = (1-m)[1-\delta+\delta v_k]$$

and

$$m = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v_{k-2}}$$

The solution to the recursion is then:

Observation 7.

$$v_{n+1} = \frac{1-\delta(1-\bar{v})-(1-u_H)^{n+1}\delta^n(1-\delta)-(1-\delta)(1-\delta(1-\bar{v}))\sum_{i=1}^{n}\delta^{i-1}(1-u_H)^i}{1-(u_H-\bar{v})\sum_{i=1}^{n+1}\delta^i(1-u_H)^{i-1}}$$

For $k \geq n+2$,

$$v_k = \begin{cases} \frac{1-\delta(1-v_n)-(1-u_L)^{\frac{k-n}{2}}\delta^{\frac{k-n}{2}-1}(1-\delta)-(1-\delta)(1-\delta(1-v_n))\sum_{i=1}^{\frac{k-n}{2}-1}(1-u_L)^i\delta^{i-1}}{1-(u_L-v_n)\sum_{i=1}^{\frac{k-n}{2}}\delta^i(1-u_L)^{i-1}} & \text{if k-n even} \\ \frac{1-\delta(1-v_{n+1})-(1-u_L)^{\frac{k-n-1}{2}}\delta^{\frac{k-n-1}{2}-1}(1-\delta)-(1-\delta)(1-\delta(1-v_{n+1}))\sum_{i=1}^{\frac{k-n-1}{2}-1}(1-u_L)^i\delta^{i-1}}{1-(u_L-v_{n+1})\sum_{i=1}^{\frac{k-n-1}{2}}\delta^i(1-u_L)^{i-1}} & \text{if k-n odd} \end{cases}$$

and
i) the sequence is strictly decreasing.
ii) $\lim_{k\to\infty} v_k = \max\{u_L, -\frac{1-\delta}{\delta}\}$.

Define the truncated sequence $\{v_k\}_{k=0}^K$ where $K-1$ is defined as the highest k such that $v_k > 0$, let the new sequence coincide with the original sequence from 0 to $K-1$ and let $v_K = 0$. The sequence is constructed such that:

Lemma 7. *Let $v \in [v_k, v_{k-1})$ for $k \geq n+2$ and let PK hold from shirking and ICL bind. Then $v^N = v$ if and only if $v^E \in [v_{k-2}, v_{k-3}]$*

Proof. The proof is analogous to that of lemma 1, □

I now define the candidate below v_n recursively.

Definition 12. *Let the sequence of function $\{\hat{F}_k\}_n^K$ satisfy the recursion: $\hat{F}_n(v) = \hat{H}_n(v)$,*

$$\hat{F}_{n+1}(v) = \frac{(1-\delta)(1-mc)+\delta m\hat{F}_n(v^E)}{1-\delta(1-m)}$$

such that

$$v = (1-m)[1-\delta+\delta v]$$

and

$$m = \frac{(1-\delta)(1-u_H)}{1-\delta+\delta v^E}$$

for $n+2$,

$$\hat{F}_{n+2} = \frac{(1-\delta)(1-mc)+\delta m\hat{F}_n(v_n)}{1-\delta(1-m)}$$

such that

$$v = (1-m)[1-\delta+\delta v]$$

and for $k > n+2$,

$$\hat{F}_k = \frac{(1-\delta)(1-mc) + \delta m \hat{F}_{k-2}(v^E)}{1-\delta(1-m)}$$

such that

$$v = (1-m)[1-\delta+\delta v]$$

and

$$m = \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E}$$

The solution to this recursion is then given by:

Observation 8.

$$\hat{F}_{n+1}(v) = v\left(\delta\sum_{i=0}^{n-2}\delta^i(1-u_H)^i + c(1-\delta)\sum_{i=0}^{n-2}(n-1-i)\delta^i(1-u_H)^i - \frac{\bar{w}}{1-\bar{v}}\sum_{i=0}^{n-1}\delta^i(1-u_H)^i\right)$$
$$+ (1-\delta)\left(\sum_{i=0}^{n-2}\delta^i(1-u_H)^i - c\left(n-1-(1-\delta)(1-u_H)\sum_{i=0}^{n-3}(n-2-i)\delta^i(1-u_H)^i\right)\right)$$
$$+ \frac{\bar{w}}{1-\bar{v}}\left(u_H+\delta(1-u_H)-(1-\delta)(1-u_H)\sum_{i=0}^{n-3}\delta^{i+1}(1-u_H)^{i+1}\right)$$

and for $k \geq n+2$,

$$\hat{F}_k(v) = A_k + B_k v$$

where for k even,

$$A_k =$$
$$-\frac{1}{(1-\delta(1-u_H))^2}\delta^{\frac{k}{2}+1}(1-(1-u_H)\delta)(u_H-u_L)(1-u_L)^{\frac{k}{2}-1}$$
$$-c(1-\delta)\left[-(1-u_H)^{n+1}\delta^{n+\frac{k}{2}}(1-u_L)^{k/2}(1-\delta)+\sum_{i=0}^{\frac{k}{2}}\left(\frac{k}{2}-i\right)\delta^i(1-u_L)^i\right.$$
$$-\sum_{i=0}^{\frac{k}{2}-2}\left(\frac{k}{2}-1-i\right)\delta^i(1-u_L)^{i+1}+n\left(\sum_{i=0}^{\frac{k}{2}}\delta^i(1-u_L)^i-\sum_{i=0}^{\frac{k}{2}-1}\delta^i(1-u_L)^{i+1}\right)$$
$$+(1-u_H)^2\delta^2\left(\frac{k}{2}-(1-\delta)\sum_{i=0}^{\frac{k}{2}-2}\left(\frac{k}{2}-1-i\right)\delta^i(1-u_L)^{i+1}+n\left(1-(1-\delta)\sum_{i=0}^{\frac{k}{2}-2}\delta^i(1-u_L)^{i+1}\right)\right)$$

$$- (1-u_H)\delta \left(k - (1-\delta)\delta^{\frac{k}{2}-1}(1-u_L)^{k/2} - (1-\delta)\sum_{i=1}^{\frac{k}{2}}(k-2i)\delta^{i-1}(1-u_L)^i \right.$$

$$\left. +2n\left(1-(1-\delta)\sum_{i=1}^{\frac{k}{2}-1}\delta^{i-1}(1-u_L)^i\right) - n(1-\delta)\delta^{\frac{k}{2}-1}(1-u_L)^{k/2}\right)\Bigg]$$

$$+ \frac{1-(1-u_H)\delta}{1-\bar{v}}\left[1-\bar{v}-\delta^{k/2}(u_H-u_L)(1-u_L)^{\frac{k}{2}-2}(1-\bar{v}-(1-u_L)(1-\bar{v}+\bar{w}))+\bar{w}u_L\right.$$

$$-(1-u_H)^n(1-\delta)\delta^{\frac{k}{2}+n}(1-u_L)^{k/2}(1-\bar{v}-(1-u_H)\bar{w})-(1-u_H)\delta(1-\bar{v}+u_L\bar{w})$$

$$\left.-\delta u_L(1-\bar{v}-(1-u_L)\bar{w})-(1-\bar{v}-(1-u_L)w)u_L(u_H-u_L)\sum_{i=0}^{\frac{k}{2}-3}\delta^{i+2}(1-u_L)^i\right]$$

and

$$B_k =$$

$$\frac{1}{(1-\delta(1-u_H))^2}v\left[\delta^{\frac{k}{2}+1}(1-(1-u_H)\delta)(u_H-u_L)(1-u_L)^{\frac{k}{2}-1}\right.$$

$$+c(1-\delta)\left(\left(1+\delta^2(1-u_H)^2\right)\sum_{i=0}^{\frac{k}{2}-1}\left(\frac{k}{2}-i\right)\delta^i(1-u_L)^i\right.$$

$$+(1-u_H)^{n+1}\delta^{n+\frac{k}{2}+1}(1-u_L)^{k/2}+n\delta^{k/2}(1-u_L)^{k/2}+n\left(1+(1-u_H)^2\delta^2\right)\sum_{i=0}^{\frac{k}{2}-1}\delta^i(1-u_L)^i$$

$$\left.-\delta(1-u_H)\left(\sum_{i=0}^{\frac{k}{2}-1}(k-2i)\delta^i(1-u_L)^i+\delta^{k/2}(1-u_L)^{k/2}+2n\sum_{i=0}^{\frac{k}{2}-1}\delta^i(1-u_L)^i+n\delta^{k/2}(1-u_L)^{k/2}\right)\right)$$

$$+\frac{1-(1-u_H)\delta}{1-\bar{v}}\left(-\bar{w}-(1-u_H)^n\delta^{n+1+\frac{k}{2}}(1-u_L)^{k/2}(1-(1-u_H)\bar{w}-\bar{v})\right.$$

$$\left.\left.+(u_H-u_L)(1-(1-u_L)\bar{w}-\bar{v})\sum_{i=0}^{\frac{k}{2}-2}(1-u_L)^i\delta^{i+2}+\delta(1-\bar{v}-(1-u_L)\bar{w}+(1-u_H)\bar{w})\right)\right]$$

and for k odd,

$$
\begin{aligned}
A_k = & \\
& - \delta^{k+1}\left(1 - \frac{1}{1-\bar{v}}\left[\bar{w}\left(-u_H\delta + \delta + u_H + (\delta-1)\sum_{i=1}^{n-2}\delta^i(1-u_H)^{i+1}\right)\right.\right. \\
& - (1-\delta)(1-\bar{v})\left(c\left(n + (\delta-1)\sum_{i=0}^{n-3}(n-2-i)\delta^i(1-u_H)^{i+1} - 1\right) - \sum_{i=0}^{n-2}\delta^i(1-u_H)^i\right) \\
& + \frac{1}{\delta^{n+1}(u_H-\bar{v})(1-u_H)^n + \delta(\bar{v}-1)+1}\left\{\left(u_H\left(-\delta^n(1-u_H)^n + \delta^{n+1}(1-u_H)^n + \delta(\bar{v}-1)+1\right)\right.\right. \\
& -(\delta-1)\delta^n(1-u_H)^n\bar{v}\Big)\left(-\delta(\bar{v}-1)\sum_{i=0}^{n-2}\delta^i(1-u_H)^i - \bar{w}\sum_{i=0}^{n-1}\delta^i(1-u_H)^i\right. \\
& \left.\left.\left.\left. + c(\delta-1)(\bar{v}-1)\sum_{i=0}^{n-2}(n-1-i)\delta^i(1-u_H)^i\right)\right\}\right]\right)(1-u_L)^k \\
& - c(1-\delta)\left(k - (1-\delta)\sum_{i=1}^{k}(k+1-i)(1-u_L)^i\delta^{i-1} + 1\right) \\
& - u_L\left(\sum_{i=0}^{k-1}(1-u_L)^i\delta^{i+1}\right)\left(1 - \frac{1}{1-\bar{v}}\left[\bar{w}\left(-u_H\delta + \delta + u_H + (\delta-1)\sum_{i=1}^{n-2}\delta^i(1-u_H)^{i+1}\right)\right.\right. \\
& - (1-\delta)(1-\bar{v})\left(c\left(n + (\delta-1)\sum_{i=0}^{n-3}(n-2-i)\delta^i(1-u_H)^{i+1} - 1\right) - \sum_{i=0}^{n-2}\delta^i(1-u_H)^i\right) \\
& + \frac{1}{\delta^{n+1}(u_H-\bar{v})(1-u_H)^n + \delta(\bar{v}-1)+1}\left\{\left(u_H\left(-\delta^n(1-u_H)^n + \delta^{n+1}(1-u_H)^n + \delta(\bar{v}-1)+1\right)\right.\right. \\
& -(\delta-1)\delta^n(1-u_H)^n\bar{v}\Big)\left(-\delta(\bar{v}-1)\sum_{i=0}^{n-2}\delta^i(1-u_H)^i - \bar{w}\sum_{i=0}^{n-1}\delta^i(1-u_H)^i\right. \\
& \left.\left.\left.\left. + c(\delta-1)(\bar{v}-1)\sum_{i=0}^{n-2}(n-1-i)\delta^i(1-u_H)^i\right)\right\}\right]\right) + 1
\end{aligned}
$$

and

$$
\begin{aligned}
B_k = & \\
& \left(1 - \frac{1}{1-\bar{v}}\left[\bar{w}\left(-u_H\delta + \delta + u_H + (\delta-1)\sum_{i=1}^{n-2}\delta^i(1-u_H)^{i+1}\right)\right.\right. \\
& - (1-\delta)(1-\bar{v})\left(c\left(n + (\delta-1)\sum_{i=0}^{n-3}(n-2-i)\delta^i(1-u_H)^{i+1} - 1\right) - \sum_{i=0}^{n-2}\delta^i(1-u_H)^i\right)
\end{aligned}
$$

$$+\frac{1}{\delta^{n+1}(u_H-\bar{v})(1-u_H)^n+\delta(\bar{v}-1)+1}\Bigg\{\Big(u_H\left(-\delta^n(1-u_H)^n+\delta^{n+1}(1-u_H)^n+\delta(\bar{v}-1)+1\right)$$

$$-(\delta-1)\delta^n(1-u_H)^n\bar{v}\Big)\left(-\delta(\bar{v}-1)\sum_{i=0}^{n-2}\delta^i(1-u_H)^i\right.$$

$$\left.-\bar{w}\sum_{i=0}^{n-1}\delta^i(1-u_H)^i+c(\delta-1)(\bar{v}-1)\sum_{i=0}^{n-2}(n-1-i)\delta^i(1-u_H)^i\right)\Bigg\}\Bigg]\Bigg)\sum_{i=0}^{k}\delta^{i+1}(1-u_L)^i$$

$$+c(1-\delta)\sum_{i=0}^{k}(k+1-i)\delta^i(1-u_L)^i$$

Then define the candidate function $\hat{F}:[0,\bar{v}]\to[0,1]$, for $v\in[v_k,v_{k-1})$ as

$$\hat{F}(v)=\begin{cases}F^P(v) & \text{if } k\geq n\\ \hat{F}_k(v) & \text{if } k>n\end{cases}$$

The following observation follows from the definition of the function:

Observation 9. *(Properties of $\hat{F}$)*
i) $\hat{F}$ is piecewise linear and concave.
ii) $\hat{F}$ is strictly increasing for $v<v_n$ and strictly decreasing for $v>v_n$ (generically).

We will show that $\hat{F}$ is the fixed point of the operator T.

The following lemmas are analogous to the arguments made when reducing the operator in the auxiliary problem and in the pure adverse selection case. Therefore I omit the proofs.

Lemma 8. *Let γ be an optimal policy at $T\hat{F}$. Then $m>0$ if and only if $s_H=0$.*

Lemma 9. *Let γ be an optimal policy at $T\hat{F}$. Then wlog $v^S=v^S_H=0$.*

Lemma 10. *Let γ be an optimal policy at $T\hat{F}$ such that $s_L=0$. Then ICL binds. Therefore $s_L=1$ wlog.*

Since the principal only monitors if $s_H=0$, ICH can be written as

$$m\geq\frac{(1-\delta)(1-u_H)}{1-\delta+\delta v^E_H}(1-s_H)$$

and the low type always shirks without loss, ICL can be written as

$$m \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E}$$

This has reduced T to

$$\begin{aligned}
T\hat{F}(v) = \max_{\gamma}\ & (1-\delta)(1-mc) + \delta(1-m)\hat{F}(v^N) + \delta m\hat{F}(v^E) \\
\text{subject to} \quad & v = (1-m)[1-\delta+\delta v^N] && (PK) \\
& m \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E} && (ICL) \\
& m \geq \frac{(1-\delta)(1-u_H)}{1-\delta+\delta v_H^E} && (ICH)
\end{aligned}$$

Claim 10. *There exists $\delta^* \in (0,1)$ such that if $\delta > \delta^*$,$T\hat{F} = \hat{F}$.*

Proof. By claim 9 for $v \in [v_n, \bar{v}]$ $F(v) = \hat{F}(v)$ so $T\hat{F} = \hat{F}$.
It remains to prove it for $v \in [0, v_n)$. Define for any policy γ

$$J(\gamma) := (1-\delta)(1-mc) + \delta(1-m)\hat{F}(v^N) + \delta m\hat{F}(v^E)$$

. Let $v \in [v_{n+1}, v_n)$. Define for any policy γ,

$$J_{n+1}(\gamma) := (1-\delta)(1-mc) + \delta(1-m)\hat{F}(v^N) + \delta m\hat{F}(v^E)$$

Then $J_{n+1}(\gamma) \geq J(\gamma)$ for any γ since $\hat{F} \leq \hat{F}_k$ for all k. Define the relaxed problem

$$\begin{aligned}
F^*_{n+1}(v) = \max_{\gamma}\ & J_{n+1}(\gamma) \\
\text{subject to} \quad & v = (1-m)(1-\delta+\delta v^N) && (PK) \\
& m \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E} && (ICL) \\
& m \geq \frac{(1-\delta)(1-u_H)}{1-\delta+\delta v^E} && (RICH)
\end{aligned}$$

This is relaxed since $J_{n+1}(\gamma) \geq J_\gamma$ and the constraint RICH is implied by ICH as $v_H^E \geq v^E$ for any $v_H^E \in v_H(v^E)$. Therefore $T\hat{F}(v) \leq F^*_{n+1}(v)$.

For any γ such that PK and RICH holds,

$$J_{n+1}(\gamma) \leq F_{n+1}(v)$$

with equality only if RICH binds. Therefore $F^*_{n+1}(v) \leq F_{n+1}(v)$. If we find a policy γ such that $v^N \in [v_{n+1}]$, $v^E \in [v_n, v_{n-1}]$, PK holds, and RICH binds then we are done since for $v^E \geq v_n$, $v_H(v^E) = v^E$, so RICH is equivalent to ICH, and ICH binds imlies ICL holds. Then γ delivers $\hat{F}_{n+1}(v)$ at $T\hat{F}(v)$, achieving the upper bound, so must be optimal and

$$T\hat{F}(v) = \hat{F}_{n+1}(v) = \hat{F}(v).$$

To construct this γ, let $v^N = v$, PK hold and RICH bind. Then by lemma 7 $v^E \in [v_n, v_{n-1}]$, and we are done.

Let $v \in [v_{n+2}, v_{n+1})$. For any policy γ, define

$$J_{n+2}(\gamma) := (1-\delta)(1-mc) + \delta(1-m)\hat{F}(v^N) + \delta m\hat{F}(v_n)$$

Define the relaxed problem

$$\begin{aligned} F^*_{n+2}(v) = &\max_{\gamma} \; J_{n+2}(\gamma) \\ \text{subject to} \quad & v = (1-m)[1-\delta+\delta v^N] \qquad (PK) \end{aligned}$$

Then $T\hat{F}(v) \leq \hat{F}^*_{n+2}(v)$. For any γ such that PK holds,

$$J_{n+2}(\gamma) = \hat{F}_{n+2}(v)$$

so $F^*_{n+2}(v) \leq \hat{F}_{n+2}(v)$. Therefore if we find a policy γ such that $v^N \in [v_{n+2}, v_{n+1}]$, $v^E = v^E_H = v_n$, ICH and ICL hold, then we are done since $T\hat{F}(v) = \hat{F}_{n+2}(v) = \hat{F}(v)$.

Construct the policy γ by setting $v_N = v$ and $v^E = v^E_H = v_n$. This is feasible since $v_H(v_n) = v_n$.

If PK holds, then m is such that

$$v = (1-m)(1-\delta+\delta v)$$

By definition of the $\{v_k\}$ sequence, if $v = v_{n+1}$ then PK implies

$$m = \frac{(1-\delta)(1-u_H)}{1-\delta+\delta v_n}$$

As we lower v from v_{n+1} in the interval $[v_{n+2}, v+n+1)$, by PK, m increases, so

$$m \geq \frac{(1-\delta)(1-u_H)}{1-\delta+\delta v_n}$$

and ICH holds.
Similarly, if $v = v_{n+2}$, by definition of the sequence,

$$m = \frac{(1-\delta)(1-u_L)}{1-\delta=\delta v_n}$$

As we increase v from v_{n+2} in the interval, PK implies m decreases, so

$$m \leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v_n}$$

and ICL holds.
Let $v \in [v_k, v_{k-1})$, $k \in \{n+3, ..., K\}$. Define

$$J_k(v) := (1-\delta)(1-mc) + \delta(1-m)\hat{F}_k(v^N) + \delta\hat{F}_{k-2}(v^E)$$

and the relaxed problem

$$\begin{aligned} F_k^*(v) &= \max_{\gamma} J_k(\gamma) \\ \text{subject to} \quad v &= (1-m)[1-\delta+\delta v^N] && (PK) \\ m &\leq \frac{(1-\delta)(1-u_L)}{1-\delta+\delta v^E} && (ICL) \end{aligned}$$

Then $T\hat{F}(v) \leq F_k^*(v)$. For any γ such that PK and ICL hold,

$$J_k(v) \leq \hat{F}_k(v)$$

with inequality only if ICL binds. Therefore if we can find a policy γ such that $v^N \in [v_k, v_{k-1}]$, $v^E \in [v_{k-2}, v_{k-3}]$, such that PK holds and ICL binds then we are done since then ICH must hold, and $T\hat{F}(v) = \hat{F}_k(v) = \hat{F}(v)$.

Construct the policy with $v^N = v$, v^E such that PK holds and ICL binds. By Lemma 7 then $v^E \in [v_{k-2}, v_{k-3}]$, and we are done.
We have proved that $T\hat{F} = \hat{F}$. □

Proof of proposition 9: The properties of F are in observation 9. We have shown that it is optimal to have the low type shirk at any value, and the high type exert effort for all $v \leq (1-\delta)u_H + \delta\bar{v}$, and that he must be allowed to shirk above this. The monitoring probability is zero in the region where the high type shirks, and satisfies the high type's incentive constraint binding and $v^E = \bar{v}$ for $v \in [v_1, (1-\delta)u_H + \delta\bar{v}]$. For any other v, the proofs of claims 8 and 10 show that it is optimal to set $v^N = v$ and since the low type shirks, PK pins down the monitoring probability as

$$m = \frac{(1-\delta)(1-v)}{1-\delta+\delta v}$$

The proofs of these claims also show that if $v \in [v_k, v_{k-1})$, it is necessary that $v^N \in [v_k, v_{k-1}]$, and $v^E \in [v_{k-1}, v_{k-2}]$ if $v \geq v_{n+1}$ and $v^E \in [v_{k-2}, v_{k-3}]$ if $v < v_{n+1}$ - this is because for any other policy the payoff in the main problem is strictly worse than the relaxed problems we defined, while in this range the payoffs coincide. It also follows from the proofs of these claims that ICH binds above v_{n+1} and ICL binds below v_{n+2}. The uniqueness of optimal continuation values when policies are on the kinks of the function follows by definition of the recursion that the kinks satisfy.

The proofs of propositions 10 and 11 are analogous to that of propositions 3 and 4, and hence omitted.

Proof Theorem 3 and Corollary 1: This follows by combining the results of propositions 8, 9 and 11. The optimal contract delivers a time zero value to the lowtype of $v_{n^*} < v_{n+1}$ and a Phase 2 value of v_{n^*-2}. This determines the sequence of decreasing monitoring probabilities until the continuation value hits $\bar{v}$. After this, high type is allowed to shirk for some periods until the value drifts below $(1-\delta)u_H + \delta\bar{v}$, and then the high type again exerts effort until he is monitored, and then he shirks again for a few periods. This cycle continues forever. Corollary 1 follows because the principal begins Phase 2 with some value strictly below v_n, which is Pareto inefficient, and after the principal has monitored some fixed number of times, the continuation value after effort hits v_n, and then remains on the Pareto frontier thereafter.

8.2 No Commitment

Proof of proposition 12:

Proof. To show that σ as defined in the proposition is an equilibrium, we need to check the incentives for each player. At time zero, the principal prefers to continue the relationship as long as

$$p_0(1-c)+(1-p_0)(\delta\bar{w}-c(1-\delta))\geq\bar{w}$$

which is true as long as

$$\delta\geq\frac{c+\bar{w}-p_0}{(c+\bar{w}-p_0(c+\bar{w})}$$

and the RHS is strictly less than 1 by assumption A1. The principal prefers to continue the relationship at any later date as long as neither S or N has occurred in the past since he knows that the agent is the high type and gets $1-c>\bar{w}$.
At time zero, the principal finds it optimal to monitor as long as

$$p_0(1-c)+(1-p_0)(\delta\bar{w}-c(1-\delta))\geq p_0(1-\delta)+\delta\bar{w}$$

which is true as long as

$$\delta\geq\frac{c}{c+p_0(1-c-\bar{w}}$$

where the RHS is again strictly less than 1 by assumption A1. If the above condition holds, then

$$1-c\geq(1-\delta)+\delta\bar{w}$$

automatically, so monitoring is a best response for the principal at any history after time zero at which neither N or S has occurred in the past.
If S or N has ever occurred in the past then the high type will shirk forever and therefore the principal finds it optimal to never monitor and end the relationship immediately.
The high type's incentives to exert effort as long as neither N or S has occurred in the past hold as $u_H>0$. If S or N has occurred in the past, the principal stops monitoring so the high type finds it strictly optimal to shirk.
The low type's finds it optimal to shirk, since if he deviates to effort he gets $u_L<0$ today, and then 0 tomorrow.
To show that this equilibrium is principal-optimal, I first show that the maximum equilibrium payoff for the principal in the relationship with the high type is $1-c$. Consider the

subgame in which the principal knows that the agent is the high type. Suppose, by contradiction, there is an equilibrium σ' of this subgame such that the principal's payoff is strictly more than $1 - c$. If the principal does not monitor with positive probability today, then the high type will shirk and the principal gets 0. f the principal does monitor with positive probability today, monitoring is a best response, so his payoff is at most $1 - c$. Therefore there exist a history in the future at some time t_1 such that $W(\sigma)|_{h^{t_1}} > 1 - c$. Now at h^{t_1}, the same argument applies, so there must be a history h^{t_2} such that $W(\sigma)|_{h^{t_2}} > 1 - c$. We can iterate this argument to construct an infinite sequence of histories, $\{h^{t_k}\}_{k=1}^{\infty}$, each of which has a payoff of at most $1 - c$ today. Since there is discounting, there is some time T such that for histories after T, the value of the payoff from these histories onwards is negligible, and therefore the principal's payoff is at most something arbitrarily close to $1 - c$, a contradiction.

Therefore in this subgame, the best equilibrium for the principal is as defined in σ. Furthermore, it leads to immediate screening so is clearly optimal from time zero too. $\square$